

# Bounds on the topology and index of minimal surfaces

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## Abstract

We prove that for every nonnegative integer  $g$ , there exists a bound on the number of ends of a complete, embedded minimal surface  $M$  in  $\mathbb{R}^3$  of genus  $g$  and finite topology. This bound on the finite number of ends when  $M$  has at least two ends implies that  $M$  has finite stability index which is bounded by a constant that only depends on its genus.

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## 1 Introduction

Let  $\mathcal{M}$  be the space of connected, properly embedded minimal surfaces in  $\mathbb{R}^3$ . The focus of this paper is to prove the existence of an upper bound on the number of ends for a surface  $M \in \mathcal{M}$  having finite topology, solely in terms of the genus of  $M$ . In the case that  $M$  has more than one end, this topological bound also produces an upper bound for the index of stability of  $M$ .

There are three classical conjectures that attempt to describe the topological types of the surfaces occurring in  $\mathcal{M}$ .

**Conjecture 1.1 (Finite Topology Conjecture I, Hoffman-Meeks)** *A non-compact orientable surface with finite genus  $g$  and a finite number of ends  $k > 2$  occurs as the topological type of an example in  $\mathcal{M}$  if and only if  $k \leq g + 2$ . A minimal surface in  $\mathcal{M}$  with finite genus and two ends has genus zero and is a catenoid.*

**Conjecture 1.2 (Finite Topology Conjecture II, Meeks-Rosenberg)** *For every positive integer  $g$ , there exists a  $\Sigma_g \in \mathcal{M}$  with one end and genus  $g$ , which is unique up to congruences and homotheties. Furthermore, if  $g = 0$ , such a  $\Sigma_g$  is a plane or a helicoid.*

**Conjecture 1.3 (Infinite Topology Conjecture, Meeks)**

*A non-compact orientable surface of infinite topology occurs as the topological type of an example in  $\mathcal{M}$  if and only if it has at most two limit ends, and when it has one limit end, then its limit end has infinite genus.*

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For a detailed discussion of these conjectures and related results, we refer the reader to the survey by Meeks and Pérez [28]. However, we make here a few brief comments on what is known concerning these conjectures and which will be used in the proof of the main theorem of this paper.

Regarding Conjecture 1.1, a theorem by Collin [9] states that if  $M \in \mathcal{M}$  has finite topology and at least two ends, then  $M$  has finite total Gaussian curvature. This result implies that such a surface is conformally a compact Riemann surface  $\overline{M}$  punctured in a finite number of points and  $M$  can be defined in terms of meromorphic data on its conformal compactification  $\overline{M}$  (Osserman [41]). Collin's Theorem reduces the question of finding topological obstructions for surfaces in  $\mathcal{M}$  of finite topology and more than one end to the question of finding topological obstructions for complete, embedded minimal surfaces of finite total curvature in  $\mathbb{R}^3$ . For example, if  $M$  is a complete, embedded minimal surface in  $\mathbb{R}^3$  with finite total curvature, genus  $g$  and  $k$  ends, then  $M$  is properly embedded in  $\mathbb{R}^3$  and the Jorge-Meeks formula [23] calculates its total curvature to be  $-4\pi(g + k - 1)$ . The first topological obstructions for complete, embedded minimal surfaces  $M$  of finite total curvature were given by Jorge and Meeks [23], who proved that if  $M$  has genus zero, then  $M$  does not have 3, 4 or 5 ends. Later this result was generalized by López and Ros [26] who proved that the plane and the catenoid are the only genus-zero minimal surfaces of finite total curvature in  $\mathcal{M}$ . At about the same time, Schoen [45] proved that a complete, embedded minimal surface of finite total curvature and two ends must be a catenoid.

The existence theory for properly embedded minimal surfaces with finite total curvature was begun by Costa [11] and by Hoffman and Meeks [18], with important theoretical advances by Kapouleas [24] and Traizet [46]. A paper by Weber and Wolf [49] makes the existence assertion in Conjecture 1.1 seem likely to hold, although their results actually fall short of giving a proof of embeddedness for their examples.

Concerning Conjecture 1.2, a theorem by Meeks and Rosenberg [35] states that the plane and the helicoid are the only properly embedded, simply connected minimal surfaces in  $\mathbb{R}^3$ . They also claimed that if  $M \in \mathcal{M}$  has finite positive genus and just one end, then it is asymptotic to the end of the helicoid and can be defined analytically in terms of meromorphic data on its conformal completion, which is a closed Riemann surface; the proof of this result was given by Bernstein and Breiner [1] and a related more general result was proved by Meeks and Pérez [27]. These theoretical results together with theorems developed by Weber and Traizet [47], by Hoffman and White [22] and by Hoffman, Weber and Wolf [21] provide the theory on which a proof of the uniqueness part of Conjecture 1.2 might be based. The existence part of Conjecture 1.2 has been recently solved by Hoffman, Traizet and White [19, 20].

In relation to Conjecture 1.3, there are two important topological obstructions for surfaces in  $\mathcal{M}$  with infinite topology: Collin, Kusner, Meeks and Rosenberg [10] proved that an example in  $\mathcal{M}$  cannot have more than two limit ends, and in [33] we proved that an example in  $\mathcal{M}$  with one limit end cannot have finite genus. This last result depends on Colding-Minicozzi theory as well as on our previous results in [32] where we presented a descriptive theorem for minimal surfaces in  $\mathcal{M}$  with two limit ends and finite genus. The study of two limit end minimal surfaces is motivated by a one-parameter family of periodic examples of genus zero discovered by Riemann [42], which are defined in terms of elliptic functions on rectangular elliptic curves. In [34] we showed that if  $M \in \mathcal{M}$  has genus zero and infinite topology, then  $M$  is one of the Riemann minimal examples, and if  $M$  has finite genus and infinite topology, then each of its two limit ends are in a natural sense asymptotic to the end of a Riemann minimal example. It is worth mentioning that Hauswirth and Pacard [16] have produced examples in  $\mathcal{M}$  with finite

genus and infinite topology.

*A priori*, one procedure to obtain surfaces in  $\mathcal{M}$  with finite genus and infinite topology might be to take limits of sequences of finite total curvature examples in  $\mathcal{M}$  with a bound on their genus but with a strictly increasing number of ends. Our results in [30, 31, 32, 33] are crucial in understanding that such sequences cannot exist, and they will lead us to a proof of the following main theorem of this manuscript.

**Theorem 1.4** *A properly embedded minimal surface in  $\mathbb{R}^3$  with finite topology has a bound on the number of its ends that only depends on its genus.*

Colding and Minicozzi [7] proved that a complete, embedded minimal surface of finite topology in  $\mathbb{R}^3$  is properly embedded. In particular, the conclusion of Theorem 1.4 remains valid if we weaken the hypothesis of properness to the hypothesis of completeness.

By a theorem of Fischer-Colbrie [14], a complete, immersed, orientable minimal surface  $M$  in  $\mathbb{R}^3$  has finite index of stability if and only if it has finite total curvature. The index of such an  $M$  is equal to the index of the Schrödinger operator  $L = \Delta + \|\nabla N\|^2$  associated to the meromorphic extension of the Gauss map  $N$  of  $M$  to the compactification of  $M$  by attaching its ends. Grigor'yan, Netrusov and Yau [15] made an in depth study of the relation between the degree of the Gauss map and the index of a complete minimal surface of finite total curvature. In particular, they proved that the index of a complete, embedded minimal surface with  $k$  ends is bounded from below by  $k - 1$ . On the other hand, Tysk [48] proved that the stability index of  $L$  can be explicitly bounded from above in terms of the degree of  $N$ . By the Jorge-Meeks formula for such an embedded  $M$ , the degree of  $N$  equals  $g + k - 1$ , where  $g$  is the genus and  $k$  is the number of ends. Hence by Theorem 1.4, if  $g$  is fixed, then  $k$  is bounded for an embedded  $M$ . Thus, one obtains the following consequence to Theorem 1.4.

**Theorem 1.5** *If  $M \subset \mathbb{R}^3$  is a complete, connected, embedded minimal surface with finite index of stability, then the index of  $M$  can be bounded by a constant that only depends on its (finite) genus. In the case of genus zero, the surface  $M$  is a plane or catenoid, and so this index upper bound is 1.*

For any integer  $k \geq 2$ , the  $k$ -noid defined by Jorge and Meeks [23] has genus zero,  $k$  catenoid type ends and index  $2k - 3$  (Montiel-Ros [40] and Ejiri-Kotani [13]). Also, there exist examples of complete, immersed minimal surfaces of genus zero with a finite number of parallel catenoidal ends but which have arbitrarily large index of stability. These examples demonstrate the necessity of the embeddedness hypothesis in Theorem 1.5.

The proof of Theorem 1.4 depends heavily on results developed in our previous papers [30, 31, 32, 33]. These papers, as well as the present one, rely on a series of deep works by Colding and Minicozzi [3, 4, 5, 6, 8] in which they describe the local geometry of a complete, embedded minimal surface in a Riemannian three-manifold, where there is a local bound on the genus of the surface.

## 2 Preliminaries.

Throughout the paper, we will denote by  $\mathbb{B}(x, r)$  the open ball in  $\mathbb{R}^3$  with center  $x \in \mathbb{R}^3$  and radius  $r > 0$ , and by  $\overline{\mathbb{B}}(x, r)$  its closure. When  $x$  is the origin, we will simply write  $\mathbb{B}(r), \overline{\mathbb{B}}(r)$  respectively for these particular open and closed balls. We let  $\mathbb{D}(r)$  denote the open disk of

radius  $r$  centered at the origin in  $\mathbb{R}^2$ . For a surface  $\Sigma \subset \mathbb{R}^3$ ,  $K_\Sigma$  will denote its Gaussian curvature function.

**Definition 2.1** Let  $A$  be an open subset of  $\mathbb{R}^3$ . A sequence of surfaces  $\{\Sigma_n\}_n$  in  $A$  is said to have *locally bounded curvature in  $A$* , if for every compact ball  $B \subset A$ , the sequence of functions  $\{K_{\Sigma_n \cap B}\}_n$  is uniformly bounded. A sequence  $\{\Sigma_n\}_n$  of properly embedded surfaces in  $A$  is called *locally simply connected*, if for every  $q \in A$ , there exist  $\varepsilon_q > 0$  and  $n_q \in \mathbb{N}$  such that for  $n > n_q$ , the components of  $\Sigma_n \cap \mathbb{B}(q, \varepsilon_q)$  are disks with their boundaries in the boundary of  $\mathbb{B}(q, \varepsilon_q)$ .

Proposition 1.1 in Colding and Minicozzi [7] ensures that the property that a sequence of embedded minimal surfaces  $\{M(n)\}_n$  is locally simply connected in  $A$  is equivalent to the property that  $\{M(n)\}_n$  has *locally positive injectivity radius in  $A$* , in the sense that for every  $q \in A$ , there exist  $\varepsilon_q > 0$  and  $n_q \in \mathbb{N}$  such that for  $n > n_q$ , the restriction to  $M(n) \cap \mathbb{B}(q, \varepsilon_q)$  of the injectivity radius function  $I_{M(n)}$  of  $M(n)$  is greater than a positive constant independent of  $n$ . In particular, if the  $M(n)$  also have boundary, then for any  $p \in A$  there exist  $\delta_p > 0$  and  $n_p \in \mathbb{N}$  such that  $\partial M(n) \cap \mathbb{B}(p, \delta_p) = \emptyset$  for all  $n > n_p$ , i.e., points in the boundary of  $M(n)$  must eventually diverge in space or converge to a subset of  $\mathbb{R}^3 - A$ .

We will call a sequence of surfaces  $\{\Sigma_n\}_n$  in  $\mathbb{R}^3$  *uniformly locally simply connected*, if there exists  $\varepsilon > 0$  such that for every  $x \in \mathbb{R}^3$ ,  $\Sigma_n \cap \mathbb{B}(x, \varepsilon)$  consists of disks with boundary in  $\partial \mathbb{B}(x, \varepsilon)$  for all  $n$  sufficiently large (depending on  $x$ ).

In the proof of our main Theorem 1.4 we will use repeatedly the following statement, which is an application of Theorem 1.6 in [31].

**Theorem 2.2** Suppose  $W$  is a closed countable subset of  $\mathbb{R}^3$ . Let  $\{M_n\}_n$  be a sequence of smooth, connected, properly embedded minimal surfaces in  $A = \mathbb{R}^3 - W$  such that:

- i.  $\{M_n\}_n$  has locally positive injectivity radius in  $A$ .
- ii. For each  $n \in \mathbb{N}$ ,  $M_n$  has compact boundary (possibly empty) and genus at most  $g \in \mathbb{N} \cup \{0\}$ , for some  $g$  independent of  $n$ .

Then, after replacing  $\{M_n\}_n$  by a subsequence and composing the surfaces with a fixed rotation, there exists a minimal lamination  $\mathcal{L}$  of  $A$  and a closed subset  $S(\mathcal{L}) \subset \mathcal{L}$  such that  $\{M_n\}_n$  converges  $C^\alpha$ , for all  $\alpha \in (0, 1)$ , on compact subsets of  $A - S(\mathcal{L})$  to  $\mathcal{L}$ ; here  $S(\mathcal{L})$  is the singular set of convergence<sup>1</sup> of the  $M_n$  to  $\mathcal{L}$ .

Furthermore:

1. The closure  $\overline{\mathcal{L}}$  of  $\mathcal{L}$  in  $\mathbb{R}^3$  has the structure of a minimal lamination of  $\mathbb{R}^3$ .
2. If  $S(\mathcal{L}) \neq \emptyset$ , then the convergence of  $\{M_n\}_n$  to  $\overline{\mathcal{L}}$  has the structure of a horizontal limiting parking garage structure in the following sense:
  - 2.1.  $\overline{\mathcal{L}}$  is a foliation of  $\mathbb{R}^3$  by horizontal planes and  $\overline{S(\mathcal{L})}$  consists of one or two vertical straight lines (called columns of the limiting parking garage structure).
  - 2.2. As  $n \rightarrow \infty$ , a pair of highly sheeted multivalued graphs are forming inside  $M_n$  around each of the lines in  $\overline{S(\mathcal{L})}$ , and if  $\overline{S(\mathcal{L})}$  consists of two lines, then these pairs of multivalued graphs inside the  $M_n$  around different lines are oppositely handed.

<sup>1</sup>This means that for every  $x \in S(\mathcal{L})$  and for all  $r > 0$ , we have  $\limsup |K_{M_n \cap \mathbb{B}(x, r)}| = \infty$ .

3. If  $\overline{\mathcal{L}}$  contains a non-flat leaf, then  $S(\mathcal{L}) = \emptyset$  and  $\overline{\mathcal{L}}$  consists of a single leaf  $L_1$ , which is properly embedded in  $\mathbb{R}^3$  and the genus of  $L_1$  is at most  $g$ . Furthermore,  $\{M_n\}_n$  converges smoothly on compact sets in  $\mathbb{R}^3$  to  $L_1$  with multiplicity 1 and one of the following three cases holds for  $L_1$ .

- (a)  $L_1$  has one end and it is asymptotic to a helicoid.
- (b)  $L_1$  has non-zero finite total curvature.
- (c)  $L_1$  has two limit ends.

*Proof.* The existence of the minimal lamination  $\mathcal{L}$  of  $A$  such that the  $M_n$  converge to  $\mathcal{L}$  (after passing to a subsequence) in  $A - S(\mathcal{L})$  follows directly from the main statement of Theorem 1.6 in [31]; note that singularities of  $\mathcal{L}$  are ruled out in our setting by item 7.1 of Theorem 1.6 in [31] because the  $M_n$  have uniformly bounded genus. The same argument using item 7.1 of Theorem 1.6 in [31] ensures that item 1 of Theorem 2.2 holds.

Now assume that  $S(\mathcal{L}) \neq \emptyset$  and we will prove that item 2 holds. To accomplish this, we first define a sequence of auxiliary compact minimal surfaces. For each  $k \in \mathbb{N}$ , choose an  $R_k \in (k, k+1)$  such that the sphere  $\partial\mathbb{B}(R_k)$  is disjoint from  $W$ , which is possible since  $W$  is a countable set. Since  $W \cap \overline{\mathbb{B}}(k+1)$  is a compact set, then  $\partial\mathbb{B}(R_k)$  is at a positive distance  $2d_k$  from  $W$ ; define  $W_k = W \cap \mathbb{B}(R_k)$  and for any  $\varepsilon > 0$  let  $W_k(\varepsilon)$  be the open  $\varepsilon$ -neighborhood of  $W_k$  in  $\mathbb{R}^3$ . As the sequence  $\{M_n\}_n$  is locally simply connected in  $A$  and  $W$  is at a positive distance from  $\partial\mathbb{B}(R_k)$ , then for  $k$  fixed and  $n$  sufficiently large,  $\partial M_n$  is disjoint from  $\partial\mathbb{B}(R_k)$  and by Sard's Theorem, we can also assume that  $R_k$  is chosen so that  $\partial\mathbb{B}(R_k)$  is transverse to all of the surfaces  $M_n$ . Also by Sard's Theorem, there exist smooth, compact, possibly disconnected three-dimensional domains  $\Delta_{n,k} \subset \mathbb{B}(R_k)$  satisfying  $W_k \subset \text{Int}(\Delta_{n,k}) \subset W_k(d_k/n)$  and such that  $\partial\Delta_{n,k}$  is transverse to  $M_n$  for all  $k$  and for  $n$  sufficiently large. It follows that  $M_{n,k} = [M_n - \text{Int}(\Delta_{n,k})] \cap \mathbb{B}(R_k)$  is a doubly indexed sequence of smooth compact surfaces such that, after replacing  $M_n$  by a subsequence,  $\{M_{k,k}\}_k$  is a sequence of smooth, compact embedded minimal surfaces that satisfy the hypotheses of Theorem 2.2. By construction, the sequence  $\{M_{k,k}\}_k$  also converges to  $\mathcal{L}$  with the same singular set of convergence  $S(\mathcal{L})$ . However, since the minimal surfaces in the sequence  $\{M_{k,k}\}_k$  are compact with genus at most  $g$ , then item 7.3 in Theorem 1.6 of [31] implies that item 2 of Theorem 2.2 holds.

Finally we demonstrate item 3 of Theorem 2.2. Suppose that the non-singular minimal lamination  $\overline{\mathcal{L}}$  contains a non-flat leaf  $L_1$ . By item 6 of Theorem 1.6 in [31],  $L_1$  is properly embedded in a simply connected domain in  $\mathbb{R}^3$ , which implies that it is two-sided. In particular,  $L_1$  is not stable and thus, the convergence of portions of the  $\{M_{k,k}\}_k$  to  $L_1$  must have multiplicity 1 (see e.g., Lemma 3 in [29]). This last property and a standard curve lifting argument ensure that the genus of  $L_1$  is at most  $g$ . By item 6 of Theorem 1.6 in [31],  $L_1$  is proper in  $\mathbb{R}^3$  and it is the unique leaf of  $\overline{\mathcal{L}}$  (that is, possibility 6.1 in that theorem holds, because case 6.2 in the same theorem cannot occur due to the finiteness of the genus of  $L_1$ ). In particular,  $S(\mathcal{L}) = \emptyset$  since through every point in  $S(\mathcal{L})$  there passes a planar leaf of  $\overline{\mathcal{L}}$  by item 5 of Theorem 1.6 in [31]). This proves the main statement in item 3 of Theorem 2.2. Item 3(a) follows from work of Bernstein and Breiner [1] or Meeks and Pérez [27]. Item 3(b) occurs when the number  $k$  of ends of  $L_1$  satisfies  $2 \leq k < \infty$ , as follows from Collin [9]. Since  $L_1$  has finite genus, then item 3(c) occurs when  $k = \infty$  by Theorem 1 in [33]. Now the proof is complete.  $\square$

**Remark 2.3** Theorem 0.9 in Colding-Minicozzi [8] can be viewed as a particular case of Theorem 2.2 when  $\{M_n\}_n$  is a locally simply connected sequence in  $\mathbb{R}^3$  of compact planar domains

with  $\partial M_n \subset \partial \mathbb{B}(R_n)$  and  $R_n \rightarrow \infty$  (hence  $W = \emptyset$ ), with the additional assumption that there exists  $R > 0$  such that each  $M_n$  intersects  $\mathbb{B}(R)$  in a component that is not a disk (this is hypothesis (0.5) in [8], which in turn allows one to discard the case of a single column for the limiting parking garage structure).

### 3 The proof of Theorem 1.4.

By Collin [9] and López-Ros [26], the catenoid is the only properly embedded, connected genus-zero minimal surface with at least two ends and finite topology, and so Theorem 1.4 holds for genus-zero surfaces. Arguing by contradiction, suppose that for some positive integer  $g$ , there exists an infinite sequence  $\{M(n)\}_{n \in \mathbb{N}}$  of properly embedded minimal surfaces in  $\mathbb{R}^3$  of genus  $g$  such that for every  $n$ , the number of ends of  $M(n)$  is finite and strictly less than the number of ends of  $M(n+1)$  and the number of ends of  $M(1)$  is at least three. By Collin's Theorem [9], each of these surfaces has finite total curvature with planar and catenoidal ends, where all ends can be assumed to be horizontal for all  $n$  after a suitable rotation.

#### 3.1 Sketch of the argument.

The argument to find the desired contradiction to prove Theorem 1.4 is based on an inductive procedure, each of whose stages starts by finding a scale of smallest non-trivial topology for the sequence of surfaces; next we will identify the limit  $L$  of the sequence in this scale (after passing to a subsequence) as being a properly embedded minimal surface in  $\mathbb{R}^3$  with controlled geometry; this control will allow us to perform a surgery on the surfaces of the sequence (for  $n$  sufficiently large) that simplifies their topology. This simplification of the topology will allow us to find another smallest scale of non-trivial topology in the subsequent stage of the process and then to repeat the arguments. The fact that all surfaces in the original sequence have fixed genus insures that after finitely many stages in the procedure, the simplification of the topology of the surfaces cannot be by lowering their genus. This fact will be used to prove that after some stage in the process, all limit surfaces  $L$  that we obtain with this procedure will be catenoids. In turn, this fact will allow us to find compact subdomains  $\Lambda(n)$  with boundary inside the  $M(n)$  (for  $n$  sufficiently large), that reproduce almost perfectly formed, large compact pieces of the limit catenoids suitably rescaled, and we will show that we can find as many of these subdomains  $\Lambda(n)$  as we like and whose associated almost waist circles  $\Gamma_n \subset \Lambda(n)$  separate  $M(n)$  and such that these domains form a pairwise disjoint collection. This separation property of the waist circles will let us control the flux vector of  $M(n)$  along  $\Gamma_n$ . The final contradiction will follow from an application the López-Ros deformation to a suitable non-compact, genus-zero subdomain in  $M(n)$  bounded by two of these separating curves  $\Gamma_n$ .

#### 3.2 Rescaling non-trivial topology in the first stage.

The asymptotic behavior of  $M(n)$  implies that for each  $n \in \mathbb{N}$ , there exists a positive number  $r_{1,n}$  such that every open ball in  $\mathbb{R}^3$  of radius  $r_{1,n}$  intersects the surface  $M(n)$  in simply connected components and there is some point  $T_{1,n} \in \mathbb{R}^3$  such that  $\mathbb{B}(T_{1,n}, r_{1,n})$  intersects  $M(n)$  in at least one component that is not simply connected. Then, the rescaled and translated minimal surfaces

$$M_{1,n} = \frac{1}{r_{1,n}}(M(n) - T_{1,n}).$$

of finite total curvature have horizontal ends and satisfy the following uniformly locally simply connected property for all  $n \in \mathbb{N}$ :

( $\star$ ) Every open ball of radius 1 intersects  $M_{1,n}$  in disk components, and the closed unit ball  $\mathbb{B}(1)$  intersects  $M_{1,n}$  in a component  $\Omega_n$  that is not simply connected.

By property ( $\star$ ), the compact semi-analytic set  $M_{1,n} \cap \overline{\mathbb{B}}(1)$  with  $M_{1,n} \cap \partial\mathbb{B}(1)$  being analytic, which admits a triangulation by [25], contains a non-simply connected component  $\Omega_n$ . It is straightforward to prove that  $\partial\Omega_n$  contains a piecewise-smooth simple closed curve  $\Gamma_n$  that does not bound a disk in  $M_{1,n} \cap \overline{\mathbb{B}}(1)$ ; for example, see [38] for similar constructions. Thus, property ( $\star$ ) implies the next one:

( $\star\star$ ) There exists a piecewise-smooth simple closed curve  $\beta(n) \subset \partial\Omega_n \subset M_{1,n} \cap \partial\mathbb{B}(1)$  that is not the boundary of a disk in  $M_{1,n}$ , where  $\Omega_n$  is defined in property ( $\star$ ).

### 3.3 Controlling the limit of the rescaled surfaces in the first stage.

By the main statement in Theorem 2.2 applied to the  $M_{1,n}$  with  $W = \emptyset$ , after passing to a subsequence, the surfaces in the sequence  $\{M_{1,n}\}_n$  converge to a minimal lamination  $\mathcal{L}_1$  of  $\mathbb{R}^3$  with related singular set of convergence  $S(\mathcal{L}_1)$ . Assume for the moment that  $S(\mathcal{L}_1) = \emptyset$  (this property will be proven in Lemma 3.4 below).

**Lemma 3.1** *If  $S(\mathcal{L}_1) = \emptyset$ , then  $\mathcal{L}_1$  consists of a single leaf  $L_1$  that is not simply connected, has genus at most  $g$  and is properly embedded in  $\mathbb{R}^3$ .*

*Proof.* By item 3 of Theorem 2.2, it suffices to prove that  $\mathcal{L}_\infty$  contains a non-simply connected leaf.

Reasoning by contradiction, suppose that  $\mathcal{L}_1$  consists entirely of leaves that are complete minimal surfaces that are simply connected. Theorem 2.2 implies that either  $\mathcal{L}_1$  has a single leaf that is non-flat and properly embedded in  $\mathbb{R}^3$  or else  $\mathcal{L}_1$  consists of planar leaves. The uniqueness of the helicoid [36] demonstrates that if  $\mathcal{L}_1$  has a single leaf  $L_1$ , then this leaf is a plane or a helicoid. By Sard's Theorem, we can take  $\delta \in (1, 2)$  such that each  $M_{1,n}$  intersects the sphere  $\partial\mathbb{B}(\delta)$  transversely. Let  $\tilde{\Omega}_n$  be the component of  $M_{1,n} \cap \overline{\mathbb{B}}(\delta)$  that contains the homotopically non-trivial simple closed curve  $\beta(n) \subset \partial\Omega_n$ , where  $\beta(n)$ ,  $\Omega_n$  are given in property ( $\star\star$ ).

By the convex hull property for minimal surfaces,  $\tilde{\Omega}_n$  is a compact minimal surface whose (smooth) boundary lies in the sphere  $\partial\mathbb{B}(\delta)$ . As the (smooth) limit  $\mathcal{L}_1$  of  $\{M_{1,n}\}_n$  is a collection of planes or it is a helicoid  $\mathcal{H}$  that is a smooth, multiplicity-one limit of  $\{M_{1,n}\}_n$ , then for  $n$  large  $\tilde{\Omega}_n$  is an almost-flat disk (in the case that  $\mathcal{L}_1$  is a collection of planes) or  $\tilde{\Omega}_n$  is a disk that is a small normal graph over its orthogonal projection to the limit helicoid  $\mathcal{H}$ . By the convex hull property for minimal surfaces, the simple closed curve  $\beta(n)$  is the boundary of a disk in  $\Omega_n \subset M_{1,n}$ , which is a contradiction that proves the lemma.  $\square$

By item 3 of Theorem 2.2, then  $\mathcal{L}_1$  consists of a single leaf  $L_1$  which is a properly embedded minimal surface in  $\mathbb{R}^3$  with genus at most  $g$ , the convergence of  $\{M_{1,n}\}_n \rightarrow L_1$  has multiplicity 1 and exactly one of the cases 3(a), 3(b) or 3(c) hold.

**Lemma 3.2** *If  $S(\mathcal{L}_1) = \emptyset$ , then case 3(c) of Theorem 2.2 cannot hold.*

*Proof.* Assume that  $L_1$  has two limit ends and we will find a contradiction. In this proof we will use the general description of such a minimal surface that is given in Theorems 1 in [32] and 8.1

in [34], as well as the terminology in those theorems. After a fixed rigid motion  $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  we may assume that

$$\Sigma = A(L_1) \quad (1)$$

has an infinite number of middle ends that are horizontal planar ends. Furthermore, there is a representative  $E \subset \Sigma$  for the top limit end of  $\Sigma$  that is conformally  $\mathbb{S}^1 \times [t_0, \infty)$  punctured in an infinite set of points  $\{e_1, e_2, \dots, e_k, \dots\}$  that correspond to the planar ends of  $E$ ; here,  $\mathbb{S}^1$  is a circle of circumference equal to the vertical component of the flux vector of  $E$ , which is the integral of the inward pointing conormal of  $E$  along its boundary. In this conformal representation of  $E$ , we also have

$$x_3(\theta, t) = t, \quad x_3(e_k) < x_3(e_{k+1}) \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k \rightarrow \infty} x_3(e_k) = \infty.$$

Given  $i \in \mathbb{N}$ , let  $t_i = \frac{1}{2}[x_3(e_i) + x_3(e_{i+1})]$ . Consider the simple closed curves

$$\gamma(i) = x_3^{-1}(\{t_i\}) \subset E, \quad \gamma(0) = x_3^{-1}(\{t_0\}) = \partial E.$$

Let  $S$  be the closed horizontal slab in  $\mathbb{R}^3$  between the heights  $t_0$  and  $t_{2g+2}$ , where  $g$  is the genus of  $M_{1,n}$ . Observe that  $\Sigma \cap S \subset E$  is a connected minimal surface whose boundary consists of the two simple closed planar curves  $\gamma(0), \gamma(2g+2)$ , and that  $\Sigma \cap S$  has  $2g+2$  horizontal planar ends.

Given  $R > 0$ , let  $\mathcal{C}(R) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq R^2\}$  be the solid cylinder of radius  $R$ . For any fixed  $\varepsilon > 0$  small, there exists an  $R_1$  large such that  $\gamma(i)$  is contained in  $\mathcal{C}(R_1)$  for  $0 \leq i \leq 2g+2$ , and  $(\Sigma \cap S) - \mathcal{C}(R_1)$  consists of  $2g+2$  annular graphs which are  $\varepsilon/2$ -close in the  $C^2$ -norm to the set of planes

$$\mathcal{P} = \{x_3^{-1}(\{x_3(e_1)\}), \dots, x_3^{-1}(\{x_3(e_{2g+2})\})\}.$$

We next transport the above structure from the limit surface  $\Sigma$  to the sequence of surfaces

$$\Sigma(n) = A(M_{1,n}), \quad n \in \mathbb{N}, \quad (2)$$

that converge smoothly to  $\Sigma$  as  $n \rightarrow \infty$  (observe that *a priori*, the  $\Sigma(n)$  might fail to have horizontal ends).

For every  $R_2 > R_1$  there exists an  $N = N(R_2) \in \mathbb{N}$  such that for  $n \geq N$ ,

$$\Sigma_S(n, R_2) = \Sigma(n) \cap S \cap \mathcal{C}(R_2)$$

is  $\varepsilon$ -close to the planar domain  $\Sigma_S(R_2) = \Sigma \cap S \cap \mathcal{C}(R_2)$  in the  $C^2$ -norm; in particular  $\Sigma_S(n, R_2)$  is also a connected planar domain for all  $n \geq N$ . By initially choosing  $\varepsilon$  sufficiently small, the following properties also hold for every  $n \geq N$ :

- (B1)  $\partial \Sigma_S(n, R_2)$  consists of  $2g+4$  simple closed curves which are arbitrarily close to  $\partial \Sigma_S(R_2)$  if  $n$  is sufficiently large.
- (B2) We can approximate the curves  $\gamma(i)$  by simple closed planar curves  $\gamma(i, n)$  in  $\Sigma_S(n, R_2) \cap x_3^{-1}(\{t_i\})$ , for all  $i = 0, \dots, 2g+2$ .
- (B3)  $\partial \Sigma_S(n, R_2)$  has two components  $\gamma(0, n), \gamma(2g+2, n)$  lying on  $\partial S$  and  $2g+2$  simple closed curves  $\alpha_1(n), \dots, \alpha_{2g+2}(n) \subset \partial \mathcal{C}(R_2)$ , which are graphs over the circle  $\partial \mathcal{C}(R_2) \cap \{x_3 = 0\}$ , ordered by their relative heights, see Figure 1.



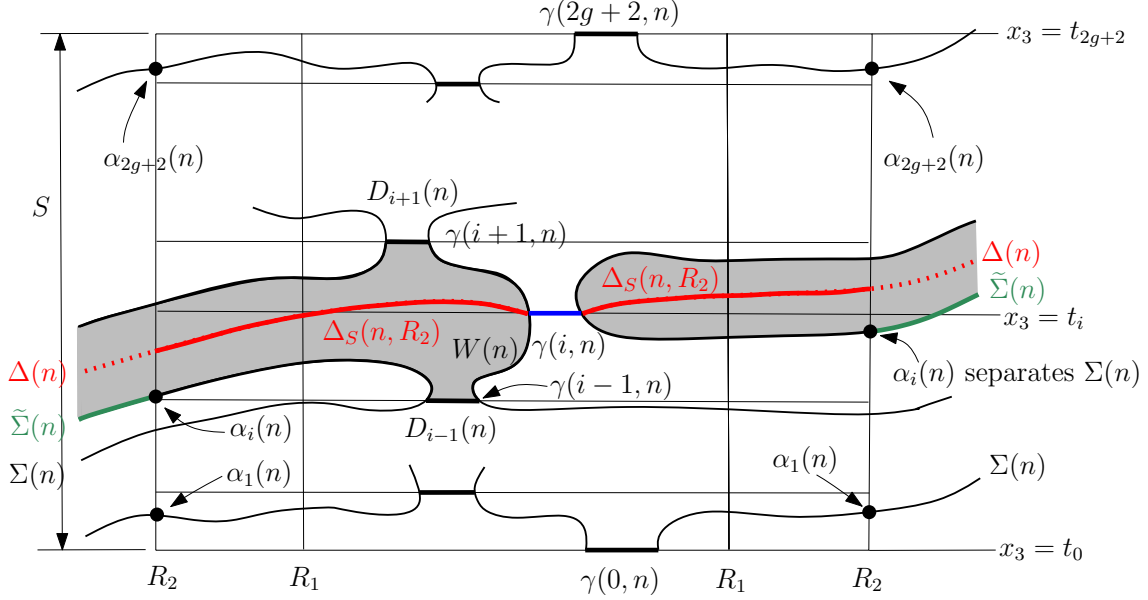


Figure 1: Producing the stable minimal surface  $\Delta(n)$  in  $W$ .

An elementary argument shows that in a compact surface with genus  $g$  and empty boundary, any connected planar subdomain with  $2g + 4$  boundary components has at least four boundary components that separate the surface. Since  $\Sigma(n)$  has genus  $g$  and  $\Sigma_S(n, R_2)$  is a connected compact planar domain with  $2g + 4$  boundary components, there exists an  $i \in \{1, \dots, 2g + 1\}$  such that the corresponding curve  $\alpha_i(n)$  separates  $\Sigma(n)$ .

**Assertion 3.3** *The simple closed curve  $\gamma(i, n)$  separates  $\Sigma(n)$ .*

*Proof of the assertion.* Let  $\tilde{\Sigma}(n)$  be the component of  $\Sigma(n) - \alpha_i(n)$  that is disjoint from  $\Sigma_S(n, R_2)$ . Note that there is a disk in  $\mathcal{C}(R_2)$  bounded by  $\alpha_i(n)$  which only intersects  $\Sigma(n)$  along  $\alpha_i(n) \cup \gamma(i, n)$ . The union of this disk with  $\tilde{\Sigma}(n)$  is a properly embedded, piecewise smooth surface in  $\mathbb{R}^3$ . After a slight perturbation of this surface in a small neighborhood of  $\tilde{\Sigma}(n)$ , we obtain a connected properly embedded surface  $\Omega(n) \subset \mathbb{R}^3$  which intersects  $\Sigma(n)$  only along  $\gamma(i, n)$ . Since properly embedded surfaces in  $\mathbb{R}^3$  separate  $\mathbb{R}^3$ , we deduce that  $\gamma(i, n)$  separates  $\Sigma(n)$ , which proves the assertion.

Note that we can assume that  $\Omega(n) - \gamma(i, n)$  consists of two components, one of which is a planar disk contained in  $S \cap \mathcal{C}(R_1)$ .  $\square$

Consider the surface  $\Omega(n)$  defined in the proof of Assertion 3.3. Let  $W(n)$  be the closed complement of  $\Sigma(n)$  in  $\mathbb{R}^3$  that intersects  $\Omega(n)$  in a non-compact connected surface with boundary  $\gamma(i, n)$ . Denote by

$$D_{i-1}(n) \subset x_3^{-1}(\{t_{i-1}\}), \quad D_{i+1}(n) \subset x_3^{-1}(\{t_{i+1}\}),$$

the planar disks bounded respectively by  $\gamma(i-1, n), \gamma(i+1, n)$ . Observe that  $\partial W(n) \cup D_{i-1}(n) \cup D_{i+1}(n)$  is a good barrier for solving Plateau problems in the abstract Riemannian piecewise smooth three-manifold  $N(n)$  obtained as the metric completion of the interior of  $W(n)$  (in particular, the planar horizontal disks  $D_{i-1}(n), D_{i+1}(n)$  could appear twice in the

boundary of  $N(n)$ ), see [39]. Also note that  $\gamma(i, n)$  separates the boundary of  $N(n)$  since the surface  $\Omega(n) \cap N(n)$  separates  $N(n)$  and  $\partial[\Omega(n) \cap N(n)] = \gamma(i, n)$ . Thus, a standard argument (see e.g., Lemma 4 in Meeks, Simon and Yau [37]) using a compact exhaustion of the closure of one of the components of  $\partial N(n) - \gamma(i, n)$  implies that we can find a connected, orientable, non-compact, properly embedded least-area surface

$$\Delta(n) \subset W(n) - [D_{i-1}(n) \cup D_{i+1}(n)] \subset \mathbb{R}^3, \text{ with } \partial\Delta(n) = \gamma(i, n).$$

By the main result in Fischer-Colbrie [14],  $\Delta(n)$  has finite total curvature and hence it has a positive finite number of planar and catenoidal ends that lie in  $W(n)$ . By the definition and the uniqueness of the limit tangent plane at infinity [2], we deduce that the limiting normal vectors to the ends of  $\Delta(n)$  are parallel to the limiting normal vectors to the ends of  $\Sigma(n)$ .

Let  $\Delta_S(n, R_2)$  be the component of  $\Delta(n) \cap S \cap \mathcal{C}(R_2)$  whose boundary contains  $\gamma(i, n)$ . Note that the other boundary components of  $\Delta_S(n, R_2)$  all lie on  $\partial\mathcal{C}(R_2) \cap W(n)$ . For  $R_1$  large and  $R_2 \gg R_1$ , curvature estimates for stable minimal surfaces [44] imply that  $\Delta_S(n, R_2) - \mathcal{C}(R_1)$  consists of almost-horizontal annular graphs, for  $n$  sufficiently large. By the area-minimizing property of  $\Delta(n)$ , we have that for  $R_2$  much larger than  $R_1$  and  $n$  sufficiently large, there is only one such almost-horizontal graph, which can be assumed to be oriented by the upward pointing normal.

Let  $\tilde{\Delta}(n)$  be the closure of  $\Delta(n) - \Delta(n, R_2)$ . Since  $\Delta(n)$  has finite total curvature, then  $\tilde{\Delta}(n)$  compactifies after attaching its ends to a compact Riemann surface  $\tilde{\tilde{\Delta}}(n)$  with boundary, and the Gauss map  $G_n: \tilde{\tilde{\Delta}}(n) \rightarrow \mathbb{S}^2$  extends smoothly across the ends to  $G_n: \tilde{\tilde{\Delta}}(n) \rightarrow \mathbb{S}^2$ , with values at the ends that lie in a pair of antipodal points  $\pm a \in \mathbb{S}^2$ . Observe that  $\tilde{\tilde{\Delta}}(n)$  is strictly stable (because  $\Delta(n)$  is stable and  $\Delta(n, R_2)$  has positive area), and so,  $\tilde{\tilde{\Delta}}(n)$  is also strictly stable. Since  $\tilde{\tilde{\Delta}}(n)$  is almost-horizontal along its boundary, then  $G_n(\partial\tilde{\tilde{\Delta}}(n))$  is contained in a small neighborhood  $Q(n, R_2)$  of  $(0, 0, 1)$  in  $\mathbb{S}^2$ . Since the Gaussian image of the ends of  $\tilde{\tilde{\Delta}}(n)$  is contained in  $\{\pm a\}$  and  $G_n$  is either constant or an open map, then we deduce that either  $G_n(\tilde{\tilde{\Delta}}(n)) \subset Q(n, R_2)$ , or else the interior of  $G_n(\tilde{\tilde{\Delta}}(n))$  contains the horizontal equator  $\mathbb{S}^2 \cap \{z = 0\}$  for  $n$  large enough. The last possibility contradicts that  $\tilde{\tilde{\Delta}}(n)$  is strictly stable, as the inner product of  $G_n$  with  $(0, 0, 1)$  is a Jacobi function on  $\tilde{\tilde{\Delta}}(n)$  whose zero set does not intersect the boundary  $\partial\tilde{\tilde{\Delta}}(n)$ . Therefore,  $G_n(\tilde{\tilde{\Delta}}(n)) \subset Q(n, R_2)$ . Note that as  $n$  and  $R_2$  approach  $\infty$ ,  $Q(n, R_2)$  limits to be  $(0, 0, 1)$ . Applying Lemma 1.4 in [35], it follows that  $\tilde{\tilde{\Delta}}(n)$  is a connected graph over its projection to the  $(x_1, x_2)$ -plane and the ends of  $\Sigma(n)$  are horizontal. This implies that the rigid motion  $A$  that appears in equations (1), (2) can be taken to be the identity map; in particular,  $L_1$  has horizontal limit tangent plane at infinity.

Since  $\gamma(i, n)$  separates  $\Sigma(n)$  (by Assertion 3.3) and the ends of  $\Sigma(n)$  are horizontal, then  $\gamma(i, n)$  has vertical flux vector, in the sense that the integral of the unit conormal vector to  $\Sigma(n)$  along  $\gamma(i, n)$  is a vertical vector. As  $\Sigma(n)$  converges  $C^2$  to  $\Sigma$  as  $n \rightarrow \infty$ , and  $\gamma(i, n)$  limits to  $\gamma(i)$ , then  $\gamma(i)$  also has vertical flux vector. But Theorem 6 in [32] implies that  $\Sigma$  does not have vertical flux vector along such a separating curve. This contradiction finishes the proof of the lemma.  $\square$

**Lemma 3.4**  $S(\mathcal{L}_1) = \emptyset$ .

*Proof.* Reasoning by contradiction, assume that  $S(\mathcal{L}_1) \neq \emptyset$ . By item 2 of Theorem 2.2,  $\mathcal{L}_1$  is a foliation of  $\mathbb{R}^3$  by parallel planes and  $S(\mathcal{L}_1)$  consists of one or two lines that are orthogonal to  $\mathcal{L}_1$ .

Suppose that  $S(\mathcal{L}_1)$  consists of a single line. By property  $(\star\star)$ ,  $M_{1,n} \cap \overline{\mathbb{B}}(1)$  contains a piecewise-smooth simple closed curve  $\beta(n)$  that does not bound a disk in  $M_{1,n}$ . We claim that the restriction of the injectivity radius function  $I_{M_{1,n}}$  of  $M_{1,n}$  to  $\beta(n)$  is bounded from below by some  $I_0 > 0$  and from above by some  $I_1 > 0$ , where both constants do not depend on  $n$ . Note that  $I_0 \geq 1$  since any open ball in  $\mathbb{R}^3$  of radius 1 intersects  $M_{1,n}$  in disks of non-positive Gaussian curvature with their boundaries in the boundary of the ball. If  $I_1$  fails to exist, then, after replacing by a subsequence, there exist points  $x_n \in \beta(n)$  such that  $I_{M_{1,n}}(x_n) > n$ . In particular, by Proposition 8 in Colding and Minicozzi [7], for  $n$  sufficiently large,  $x_n$  is contained in a component of  $M_{1,n} \cap \mathbb{B}(x_n, 3)$  that is an open disk  $D_n$  with boundary in the boundary of  $\mathbb{B}(x_n, 3)$ . By the triangle inequality, the boundary of  $\mathbb{B}(x_n, 3)$  lies outside of  $\overline{\mathbb{B}}(1)$ . Therefore, the simple closed curve  $\beta(n) \subset D_n \subset M_{1,n}$  is the boundary of a disk in  $M_{1,n}$ , which is a contradiction. This contradiction proves the claim that the values of  $I_{M_{1,n}}$  restricted to  $\beta(n)$  lie in an interval  $[I_0, I_1]$ .

A standard argument (see, for instance, Proposition 2.12, Chapter 13 of [12]) produces, for each  $n \in \mathbb{N}$ , a non-trivial geodesic loop  $\Gamma_n \subset M_{1,n}$  parameterized by arc length, possibly not smooth at  $\Gamma_n(0)$ , such that  $\Gamma_n(0) = \Gamma_n(2d_n) \in \beta(n)$  (here  $2d_n = 2I_{M_{1,n}}(\Gamma_n(0))$  is the length of  $\Gamma_n$ ), and both arcs  $\Gamma_n|_{[0, d_n]}$ ,  $\Gamma_n|_{[d_n, 2d_n]}$  minimize length among curves in  $M_{1,n}$  with their extrema. Hence, the length of  $\Gamma_n$  lies in the interval  $[2I_0, 2I_1]$ . We claim that the limit set  $\text{Lim}(\{\Gamma_n\}_n)$  satisfies  $\text{Lim}(\{\Gamma_n\}_n) \subset S(\mathcal{L}_1)$ . If on the contrary there exists a point  $x \in \text{Lim}(\{\Gamma_n\}_n) - S(\mathcal{L}_1)$ , then the  $\Gamma_n$  converge locally around  $x$  (after extracting a subsequence) to an open straight line segment  $\Gamma(x)$  passing through  $x$  and contained in one of the planes of the limit foliation  $\mathcal{L}_1$ . As we are assuming that  $S(\mathcal{L}_1)$  consists of a single line, then the straight line that contains  $\Gamma(x)$  cannot intersect  $S(\mathcal{L}_1)$  at two points. This contradicts that the length of  $\Gamma_n$  is bounded from above by  $2I_1$ , and proves our claim that  $\text{Lim}(\{\Gamma_n\}_n) \subset S(\mathcal{L}_1)$ . In particular, there exist points  $a_n, b_n \in \Gamma_n$  at intrinsic distance in  $M_{1,n}$  bounded away from zero, such that  $|a_n - b_n| \rightarrow 0$ . This contradicts the chord arc bound given by Theorem 0.5 of [7]. This contradiction proves that  $S(\mathcal{L}_1)$  cannot consist of a single line.

Since  $S(\mathcal{L}_1)$  consists of two disjoint lines  $l_1, l_2$  that are orthogonal to  $\mathcal{L}_1$ , then item 2.2 of Theorem 2.2 implies that the pairs of multivalued graphs forming inside the  $M_{1,n}$  around  $l_1, l_2$  are oppositely handed. From this point on, the proof of this lemma is similar to the proof of Lemma 3.2. We now outline the argument along the lines of the previous proof.

Recall that the Riemann minimal examples  $\{\mathcal{R}_t\}_{t>0}$  form a one-parameter family of properly embedded, singly periodic, minimal planar domains in  $\mathbb{R}^3$ , each one with infinitely many horizontal planar ends (see e.g., Section 2 of [34] for a precise description of these classical surfaces). To identify the natural limit objects for each of the two ends of this one-parameter family, one normalizes each  $\mathcal{R}_t$  suitably: after a certain normalization, the surfaces  $\mathcal{R}_t$  converge as  $t \rightarrow 0$  to a vertical catenoid, while as  $t \rightarrow \infty$ , one can normalize the  $\mathcal{R}_t$  so that under two different sequences of translations, each translated sequence converges to a vertical helicoid, with the two forming helicoids inside the  $\mathcal{R}_t$  for  $t$  large being symmetric by reflection in the vertical plane of symmetry of  $\mathcal{R}_t$ ; hence these helicoids have opposite handedness. This last property implies that after shrinking the  $\mathcal{R}_t$  suitably and taking  $t \rightarrow \infty$ , the  $\mathcal{R}_t$  limit to a foliation  $\mathcal{F}$  of  $\mathbb{R}^3$  by horizontal planes, with singular set of convergence being two vertical lines. With our language in item 2 of Theorem 2.2, this limit configuration is a limiting parking garage structure with two oppositely oriented columns; see Figure 2.

In our current setting that the  $\{M_{1,n}\}_n$  converge to the foliation  $\mathcal{L}_1$  except at two lines  $l_1, l_2$  with opposite handedness, the convergence of  $\{M_{1,n}\}_n$  to  $\mathcal{L}_1$  has the same basic structure as a

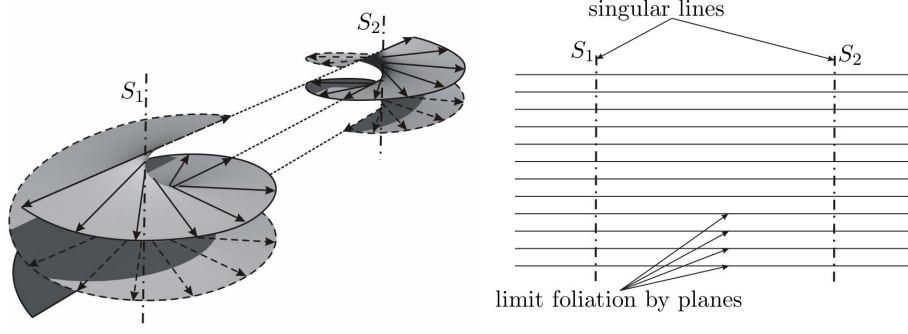


Figure 2:  $S_1$  and  $S_2$  are the columns of the limiting parking garage structure.

two-limit-end example. More precisely, let  $\mathcal{C}(R_1)$  be a solid cylinder of radius  $R_1$  that contains  $S(\mathcal{L}_1)$  in its interior. Consider the intersection of  $M_{1,n}$  with an open slab  $S$  bounded by two planes in  $\mathcal{L}_1$ . For  $R_2 > R_1$  and  $n$  sufficiently large, the part of  $M_{1,n} \cap S$  in  $\mathcal{C}(R_2) - \mathcal{C}(R_1)$  contains an arbitrarily large number of annular graphs with boundary curves in the cylinders  $\partial\mathcal{C}(R_1), \partial\mathcal{C}(R_2)$  and these graphs are almost-parallel to the planes in  $\mathcal{L}_1$ ; note that  $R_1$  can be taken large enough so that the columns of the limiting parking garage structure are contained in  $\mathcal{C}(R_1/2)$ . The remaining components of  $M_{1,n} \cap S \cap [\mathcal{C}(R_2) - \mathcal{C}(R_1)]$  are graphical almost-horizontal disk components that are contained in arbitrarily small neighborhoods of  $\partial S$  as  $n$  tends to infinity. Furthermore, inside  $\mathcal{C}(R_1) \cap S$  we can find as many of the related curves  $\gamma(i, n) \subset M_{1,n}$  from the proof of Lemma 3.2 as we desire (take the  $\gamma(i, n)$  as ‘connection loops’ that converge to a straight line segment joining  $l_1, l_2$  and orthogonal to these lines). Carrying out the arguments in the proof of the Lemma 3.2 we obtain a contradiction, which proves Lemma 3.4.  $\square$

So far we have proven that, after passing to a subsequence, the surfaces  $M_{1,n}$  converge smoothly with multiplicity 1 to a connected, properly embedded minimal surface  $L_1$  in one of the following two cases.

- (C1)  $L_1$  is a one-ended surface with positive genus less than or equal to  $g$  and  $L_1$  is asymptotic to a helicoid.
- (C2)  $L_1$  has finite total curvature, genus at most  $g$  and at least two ends.

### 3.4 Surgery in the first stage.

We start this section with a lemma to be used later.

**Lemma 3.5** *Let  $\Sigma \subset \mathbb{R}^3$  be a properly embedded minimal surface of finite genus and one end. Then, for  $R > 0$  sufficiently large,  $\partial\mathbb{B}(R)$  intersects  $\Sigma$  transversely in a simple closed curve and  $\Sigma - \mathbb{B}(R)$  is an annular end representative of the unique end of  $\Sigma$ . Furthermore, given  $\varepsilon > 0$ ,  $R$  can be chosen large enough so that this end representative is  $\varepsilon$ -close in the  $C^2$ -norm to an end of a helicoid.*

*Proof.* Since arbitrarily small perturbations of the square of the distance function of  $\Sigma$  to the origin in  $\mathbb{R}^3$  can be chosen to have only non-degenerate critical points of index less than 2, elementary Morse theory implies that for the first sentence of the lemma to hold it suffices to

prove that for  $R > 0$  sufficiently large,  $\partial\mathbb{B}(R)$  intersects  $\Sigma$  transversely. Since  $\Sigma$  is a properly embedded minimal surface of finite genus and one end, then  $\Sigma$  is asymptotic to a helicoid (Bernstein and Breiner [1] or Meeks and Pérez [27]); in particular, after a rotation in  $\mathbb{R}^3$  we may assume that for some  $R > 0$  large and  $\delta > 0$ , the intersection of  $\Sigma - \mathbb{B}(R)$  with the region

$$C(\delta) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > \delta^2 x_3^2\}$$

consists of two multivalued graphs over their projections to the  $(x_1, x_2)$ -plane with norm of their gradients less than 1, and given  $\varepsilon > 0$ ,  $R$  can be taken sufficiently large so that the norm of the gradients of these multivalued graphs are less than  $\varepsilon$ . Furthermore, the analytic description in [17, 27] ensures that there exists a conformal parameterization  $X: D(\infty, \rho) \rightarrow \mathbb{R}^3$  of an end representative of the end of  $\Sigma$ , with associated Weierstrass data

$$g(z) = e^{iz+f(z)}, \quad dh = dz, \quad (3)$$

where  $D(\infty, \rho) = \{z \in \mathbb{C} \mid |z| \geq \rho\}$ ,  $f$  is a holomorphic function in  $D(\infty, \rho)$  that extends across  $z = \infty$  with  $f(\infty) = 0$ ,  $g$  is the stereographically projected Gauss map of  $X$  and  $dh$  denotes its height differential. This analytic description implies that the following properties hold:

- (D1)  $[\Sigma - \mathbb{B}(R)] \cap C(\delta)$  is transversal to  $\partial\mathbb{B}(R')$  for every  $R' > R$ .
- (D2)  $\Sigma - \mathbb{B}(R)$  is transverse to every horizontal plane  $\{x_3 = t\}$ , for all  $t \in \mathbb{R}$ .
- (D3) For  $|t|$  sufficiently large,  $\beta_t := [\Sigma - \mathbb{B}(R)] \cap \{x_3 = t\}$  consists of a smooth, proper Jordan arc at distance less than 1 to the axis of the helicoid  $H$  to which  $\Sigma$  is asymptotic, and the tangent lines to  $\beta_t$  are arbitrarily close to the constant value determined by the intersection straight line  $H \cap \{x_3 = t\}$ .

Property (D3) ensures that circles in  $\{x_3 = t\}$  centered at  $(0, 0, t)$  and radii  $r > R_0$  are transverse to  $\beta_t$  for  $|t|, R_0$  sufficiently large. This implies that there exists  $R_1 > R_0$  large such that for  $R$  sufficiently large,  $\partial\mathbb{B}(R)$  is transverse to  $\Sigma - \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq R_1^2\}$ . Finally, transversality of  $\partial\mathbb{B}(R)$  to  $\Sigma \cap \{x_1^2 + x_2^2 \leq R_1^2\}$  for  $R$  large follows from the fact that under any divergent sequence of vertical translations of  $\Sigma$ , a subsequence converges to some vertical translation of  $H$  and, as  $R \rightarrow \infty$ , the unit normal vectors to  $\partial\mathbb{B}(R)$  at points of  $\Sigma \cap \partial\mathbb{B}(R) \cap \{x_1^2 + x_2^2 \leq R_1^2\}$  are converging to vertical unit vectors. This finishes the proof of the first sentence of the statement lemma. The second sentence follows from the Weierstrass representation equation (3).  $\square$

Given  $R_1 > 0$ , let  $L_1(R_1) = L_1 \cap \overline{\mathbb{B}}(R_1)$ . By Lemma 3.5, we can take  $R_1$  sufficiently large so that in case (C1) above,  $L_1 - L_1(R_1)$  consists of an annular end representative of the unique end of  $L_1$ , and this annular end representative is  $\varepsilon$ -close in the  $C^2$ -norm to an end of a helicoid  $H \subset \mathbb{R}^3$  for any fixed  $\varepsilon > 0$  small. In case (C2), the asymptotic geometry of embedded ends with finite total curvature allows us to assume that  $L_1 - L_1(R_1)$  consists of a finite collection of annular end representatives of the at least two ends of  $L_1$ , each one  $\varepsilon$ -close in the  $C^2$ -norm to the end of a plane or of a catenoid.

We will perform the following replacement of  $L_1(R_1)$ :

- (E1) Suppose that  $L_1$  is in case (C1) above. Replace  $L_1(R_1)$  by a smooth disk that is a small normal graph over its projection to the helicoid  $H$ , and so that the union of this new piece

with  $L_1 - L_1(R_1)$  produces a smooth, properly embedded surface  $\tilde{L}_1 \subset \mathbb{R}^3$ . For  $R_1$  large, this replacement can be made in such a way that  $\tilde{L}_1$  is  $\varepsilon$ -close to  $H$  in the  $C^2$ -norm on compact sets of  $\mathbb{R}^3$  for an arbitrarily small  $\varepsilon > 0$  (after choosing  $R_1$  sufficiently large).

(E2) Now assume  $L_1$  is in case (C2). Then, for  $R_1$  large,  $L_1 - L_1(R_1)$  consists of a finite number  $r \geq 2$  of non-compact, annular minimal graphs over the limit tangent plane at infinity of  $L_1$ , bounded by  $r$  closed curves which are almost-parallel and logarithmically close in terms of  $R$  to an equator on  $\partial\mathbb{B}(R_1)$ . Replace  $L_1(R_1)$  by  $r$  almost-flat parallel disks contained in  $\mathbb{B}(R_1)$  so that the resulting surface, after gluing these disks to  $L_1 - L_1(R_1)$ , is a possibly disconnected, smooth, properly embedded surface  $\tilde{L}_1 \subset \mathbb{R}^3$ . This replacement is made so that  $\tilde{L}_1 \cap \mathbb{B}(R_1)$  has arbitrarily small second fundamental form (after choosing  $R_1$  sufficiently large).

In either of the cases (E1), (E2), note that the surface  $\tilde{L}_1$  is no longer minimal, but it is minimal outside  $\mathbb{B}(R_1)$ .

We finish this first stage by performing the surgery on the surfaces  $M_{1,n}$  that converge to  $L_1$ . For  $n$  large, the surface  $M_{1,n}(R_1) = M_{1,n} \cap \mathbb{B}(R_1)$  can be assumed to be arbitrarily close to  $L_1(R_1)$  in the  $C^2$ -norm. Modify  $M_{1,n}$  in  $\mathbb{B}(R_1)$  as we did for  $L_1$  to obtain a new smooth, properly embedded surface  $\tilde{M}_{1,n}$  that is  $C^2$ -close to  $\tilde{L}_1$  in  $\mathbb{B}(R_1)$ . Observe that if  $L_1$  is in case (C1), then  $\tilde{M}_{1,n}$  is connected and has the same number of ends as  $M_{1,n}$ , while if  $L_1$  is in case (C2), then  $\tilde{M}_{1,n}$  might fail to be connected, and the total number of ends of the connected components of  $\tilde{M}_{1,n}$  is equal to the number of ends of  $M_{1,n}$  (see Figure 3 for a topological representation of this surgery procedure).

### 3.5 Rescaling by non-trivial topology in the second stage.

Since the number of ends of  $\tilde{M}_{1,n}$  is unbounded as  $n \rightarrow \infty$ , then  $\tilde{M}_{1,n}$  has some component that is not simply connected for  $n$  large. As  $\tilde{M}_{1,n}$  has catenoidal or planar ends, then for  $n$  large there exists a largest positive number  $r_{2,n}$  such that for every open ball  $B$  in  $\mathbb{R}^3$  of radius  $r_{2,n}$ , every simple closed curve in  $\tilde{M}_{1,n} \cap B$  bounds a disk in  $\tilde{M}_{1,n}$  (but not necessarily inside  $B$  as  $\tilde{M}_{1,n}$  does not necessarily satisfy the convex hull property), and there exists a closed ball of radius  $r_{2,n}$  centered at a point  $T_{2,n} \in \mathbb{R}^3$  whose intersection with  $\tilde{M}_{1,n}$  contains a simple closed curve that is homotopically non-trivial in  $\tilde{M}_{1,n}$ .

Since for  $n$  large, every component of  $\tilde{M}_{1,n} \cap \mathbb{B}(2R_1)$  is simply connected and we can assume  $R_1 \geq 2$ , then every simple closed curve homotopically non-trivial on  $\tilde{M}_{1,n}$  that is contained in a ball of radius 2 is necessarily disjoint from  $\mathbb{B}(R_1)$ . By the last paragraph, there exists a simple closed curve  $\Gamma \subset \tilde{M}_{1,n}$  that is homotopically non-trivial in  $\tilde{M}_{1,n}$  and that is contained in a closed ball of radius  $r_{2,n}$ . We next check that  $r_{2,n} \geq 1$ . It clearly suffices to show that this inequality holds assuming that  $r_{2,n} \leq 2$ ; in this case, the previous argument implies that  $\mathbb{B}(T_{2,n}, r_{2,n})$  does not intersect  $\mathbb{B}(R_1)$ , and so,  $\Gamma \subset M_{1,n} \cap \mathbb{B}(T_{2,n}, r_{2,n})$ . Note that  $\Gamma$  is homotopically non-trivial in  $M_{1,n}$ . In particular,  $r_{2,n} \geq 1$  by property  $(\star)$ , as desired.

Let  $\widehat{M}_{2,n} = \frac{1}{r_{2,n}}(\tilde{M}_{1,n} - T_{2,n})$ . Thus,  $\mathbb{B}(1)$  contains a simple closed curve in  $\widehat{M}_{2,n}$  that is homotopically non-trivial in  $\widehat{M}_{2,n}$ . Also define

$$M_{2,n} = \frac{1}{r_{2,n}}(M_{1,n} - T_{2,n}), \quad B_{1,n} = \frac{1}{r_{2,n}}(\mathbb{B}(R_1) - T_{2,n}). \quad (4)$$

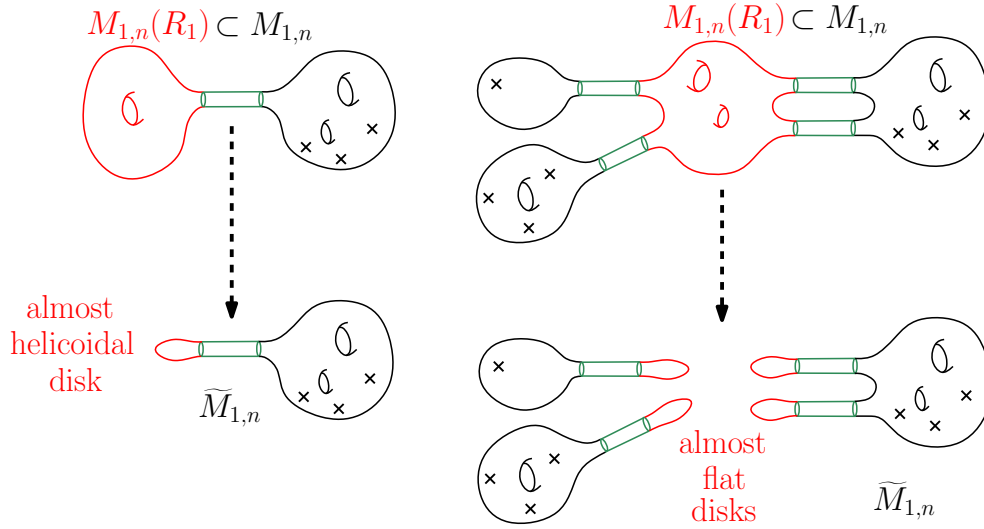


Figure 3: Topological representation of the surgery in the first stage, where the ends are represented by crosses. On the left, replacement (E1) is done on  $M_{1,n}$ , thus the compact surface  $M_{1,n}(R_1)$  (above, red) has only one boundary component and is replaced by an almost helicoidal disk (below, red); after smoothing along a transition region (represented by a green annulus), the resulting surface  $\widetilde{M}_{1,n}$  is connected and has the same number of ends as  $M_{1,n}$ . On the right, replacement (E2) occurs; in this example, the limit surface  $L_1$  has genus 2 and 4 ends, each end producing a boundary component of  $M_{1,n}(R_1)$  that is glued after the surgery to an almost flat disk. The resulting surface  $\widetilde{M}_{1,n}$  disconnects into three components, with 1, 3 and 3 ends respectively. Observe that replacement (E1) strictly decreases the genus of  $\widetilde{M}_{1,n}$ .

Clearly,  $\widehat{M}_{2,n}$  is homeomorphic to  $\widehat{M}_{1,n}$  and has simpler topology than  $M_{2,n}$ , in the sense that  $\widehat{M}_{1,n}$  has less generators for its first homology group than  $M_{2,n}$ ; the simplification of the topology of  $M_{2,n}$  giving  $\widehat{M}_{2,n}$  as a replacement of a subdomain by disks, only occurs inside the ball  $B_{1,n}$ .

### 3.6 Controlling the limit of the rescaled surfaces in the second stage.

**Lemma 3.6** *Let  $C \subset \mathbb{R}^3$  be any compact set. Then, for  $n$  large, the ball  $B_{1,n}$  is disjoint from  $C$ .*

*Proof.* Suppose to the contrary, that after passing to a subsequence, every  $B_{1,n}$  intersects a compact set  $C \subset \mathbb{R}^3$ . We first show that the radii of  $B_{1,n}$  go to zero as  $n \rightarrow \infty$ . If not, and again after taking a subsequence, we can assume that the radius of  $B_{1,n}$  is larger than some  $\varepsilon > 0$  for every  $n \in \mathbb{N}$ . Since the distance from  $\overline{\mathbb{B}}(1)$  to  $B_{1,n}$  is bounded independently of  $n$ , there exists  $r_0 > 0$  such that the open ball  $B'_{1,n}$  concentric with  $B_{1,n}$  of radius  $r_0$ , contains  $\overline{\mathbb{B}}(1)$  for every  $n$ . By our normalization and possibly applying Lemma 3.5, for  $n$  sufficiently large,  $B'_{1,n}$  intersects  $\widehat{M}_{2,n}$  in disks, which contradicts that  $\overline{\mathbb{B}}(1)$  contains a closed curve that is homotopically non-trivial in  $\widehat{M}_{2,n}$  (we are using here that our previous sequence  $\{M_{1,n}\}_n$  converges smoothly on arbitrarily large compact subsets of  $\mathbb{R}^3$  to  $L_1$  and outside  $\mathbb{B}(R_1)$ ,  $L_1$  consists of its annular ends). This contradiction shows that the radii of the balls  $B_{1,n}$  tend to zero as  $n \rightarrow \infty$ , provided that these balls intersect  $C$ .

By the previous paragraph, after taking a subsequence we can assume that the sequence of balls  $\{B_{1,n}\}_n$  converges to a point  $p \in C$ . To proceed with the proof of Lemma 3.6, we need the following assertion.

**Assertion 3.7**  $\{M_{2,n}\}_n$  is locally simply connected in  $\mathbb{R}^3 - \{p\}$ .

*Proof.* Fix a point  $q \in \mathbb{R}^3 - \{p\}$ . Then we can write  $|p - q| = d\varepsilon$  for  $d \geq 10$ ,  $\varepsilon > 0$ . Reasoning by contradiction, suppose that for  $\varepsilon$  arbitrarily small, for  $n$  large, we find a simple closed curve  $\Gamma_n \subset M_{2,n} \cap \mathbb{B}(q, \varepsilon)$  that is homotopically non-trivial in  $M_{2,n}$ . Since  $\varepsilon$  can be assumed to be less than 1, then  $\Gamma_n$  must bound a disk  $D_n$  in  $\widehat{M}_{2,n}$ . By the convex hull property,  $D_n$  is not minimal and so it must intersect  $B_{1,n}$ . Note that  $D_n - B_{1,n}$  contains a compact, connected, minimal planar domain whose boundary intersects each of the spheres  $\partial\mathbb{B}(q, \varepsilon)$  and  $\partial B_{1,n}$ . An elementary application of the maximum principle for minimal surfaces shows that there is no connected minimal surface whose boundary is contained in two such spheres (pass a suitable catenoid between the closed balls  $\overline{\mathbb{B}}(q, \varepsilon)$  and  $\overline{B_{1,n}}$ ). Thus,  $\{M_{2,n}\}_n$  is locally simply connected in  $\mathbb{R}^3 - \{p\}$ .  $\square$

We next continue with the proof of Lemma 3.6. Consider the sequence of compact minimal surfaces with boundary

$$\{M_{2,n} \cap [\overline{\mathbb{B}}(n) - \mathbb{B}(p, 1/n)]\}_n.$$

As we have already observed, this sequence has locally positive injectivity radius in  $\mathbb{R}^3 - \{p\}$ . By Theorem 2.2 applied to this sequence with  $W = \{p\}$ , there a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{p\}$  and a closed subset  $S(\mathcal{L}) \subset \mathcal{L}$  such that  $\{M_{2,n} \cap [\overline{\mathbb{B}}(n) - \mathbb{B}(p, 1/n)]\}_n$  converges  $C^\alpha$  (for all  $\alpha \in (0, 1)$ ) to  $\mathcal{L}$  outside the singular set of convergence  $S(\mathcal{L})$ . Furthermore, the closure  $\overline{\mathcal{L}}$  of  $\mathcal{L}$  in  $\mathbb{R}^3$  has the structure of a minimal lamination of  $\mathbb{R}^3$ .

**Assertion 3.8**  $\overline{\mathcal{L}}$  consists entirely of planes.



*Proof.* Reasoning by contradiction, assume that  $\overline{\mathcal{L}}$  contains a non-flat leaf. By item 3 of Theorem 2.2,  $S(\mathcal{L}) = \emptyset$  and  $\overline{\mathcal{L}}$  consists of a single leaf  $L$ , which is a properly embedded minimal surface in  $\mathbb{R}^3$  with genus at most  $g$ . Furthermore, the convergence of portions of the surfaces  $M_{2,n} \cap [\overline{\mathbb{B}}(n) - \mathbb{B}(p, 1/n)]$  to  $L$  is of multiplicity 1.

Since for all fixed  $\varepsilon > 0$  and for  $n$  large the area of  $M_{2,n} \cap \mathbb{B}(p, \varepsilon)$  is greater than  $\frac{3}{2}\pi\varepsilon^2$  (by the monotonicity of area formula) and  $S(\mathcal{L}) = \emptyset$ , then we deduce that  $p \in L$ . As  $L$  is embedded, the area of  $L \cap \mathbb{B}(p, \varepsilon)$  divided by  $\varepsilon^2$  tends to  $\pi$  as  $\varepsilon \rightarrow 0$ , which contradicts that the  $M_{2,n}$  converge smoothly to  $L$  in  $\mathbb{B}(p, \varepsilon) - \{p\}$  with multiplicity 1. This completes the proof of the assertion.  $\square$

By Assertion 3.8,  $\overline{\mathcal{L}}$  consists entirely of planes, which after a rotation in  $\mathbb{R}^3$  can be assumed to be horizontal.

Since  $\widehat{M}_{2,n} - \mathbb{B}(p, 1/2)$  is minimal for  $n$  large and every simple closed curve in  $\widehat{M}_{2,n} \cap \mathbb{B}(p, 1/2)$  is the boundary of a disk in  $\widehat{M}_{2,n}$ , then the components of  $\widehat{M}_{2,n} \cap \mathbb{B}(p, 1/2)$  are disks when  $n$  is large. It follows that  $\widehat{M}_{2,n} \cap \overline{\mathbb{B}}(1)$  contains a homotopically non-trivial simple closed curve, such that after an isotopy of such a curve in  $\widehat{M}_{2,n}$ , there is another simple closed curve  $\Gamma_n \subset \widehat{M}_{2,n} \cap \overline{\mathbb{B}}(2)$  which is homotopically non-trivial in  $\widehat{M}_{2,n}$  and disjoint from  $\mathbb{B}(p, 1/2)$ .

**Assertion 3.9** *The singular set of convergence  $S(\mathcal{L})$  of the  $M_{2,n} \cap [\overline{\mathbb{B}}(n) - \mathbb{B}(p, 1/n)]$  to  $\mathcal{L}$  is non-empty and intersects  $\mathbb{B}(5)$ .*

*Proof.* Arguing by contradiction, suppose  $S(\mathcal{L}) \cap \mathbb{B}(5) = \emptyset$ . Note that  $\mathbb{B}(p, 1/2)$  cannot intersect both of the spheres  $\partial\mathbb{B}(3)$ ,  $\partial\mathbb{B}(4)$ ; in what follows we will assume that  $\mathbb{B}(p, 1/2) \cap \partial\mathbb{B}(3)$ , as the argument in the other case is similar. Since all leaves of  $\overline{\mathcal{L}}$  are horizontal planes,  $S(\mathcal{L})$  is empty and the surfaces  $M_{2,n} \cap [\overline{\mathbb{B}}(3) - \mathbb{B}(p, 1/2)]$  are compact, then for  $n$  large, each component of  $M_{2,n} \cap [\overline{\mathbb{B}}(3) - \mathbb{B}(p, 1/2)]$  that intersects  $\overline{\mathbb{B}}(2)$  is an almost-horizontal graph over the  $(x_1, x_2)$ -plane and is either a disk with boundary in  $\partial\mathbb{B}(3)$  or an annulus with boundary in  $\partial\mathbb{B}(3) \cup \partial\mathbb{B}(p, 1/2)$  (here we may assume  $\partial\mathbb{B}(p, 1/2)$  is transverse to every such minimal surface). In particular, for  $n$  large, every component of  $\widehat{M}_{2,n} \cap \overline{\mathbb{B}}(3)$  is a disk, which contradicts the existence of the curve  $\Gamma_n$ . Thus,  $S(\mathcal{L}) \neq \emptyset$ .  $\square$

By Assertion 3.9 and item 2 of Theorem 2.2,  $\overline{\mathcal{L}}$  is a foliation of  $\mathbb{R}^3$  by (horizontal) planes and the convergence of the  $M_{2,n} \cap [\overline{\mathbb{B}}(n) - \mathbb{B}(p, 1/n)]$  to  $\overline{\mathcal{L}}$  has the structure of a horizontal limiting parking garage structure with one or two columns (vertical lines) as singular set of convergence. Note that if  $\overline{S(\mathcal{L})}$  is disjoint from  $\mathbb{B}(3)$ , then one can also apply the arguments in the proof of Assertion 3.9 to obtain a contradiction. Thus, we may assume that  $\overline{S(\mathcal{L})}$  intersects  $\mathbb{B}(3)$ . The proof of Lemma 3.4 applies to show that  $\overline{S(\mathcal{L})}$  does not contain two components (observe that the presence of the point  $p$  does not affect the argument of the proof of Lemma 3.4, as this argument can be done in a horizontal slab far from  $p$ ). Hence,  $\overline{S(\mathcal{L})}$  consists of a single line.

We now check that the possibility that  $\overline{S(\mathcal{L})}$  is a single line also leads to a contradiction, which will finish the proof of Lemma 3.6. Define  $\varepsilon > 0$  by

$$\varepsilon = \begin{cases} \min\{\frac{1}{4}, \frac{1}{2}d_{\mathbb{R}^3}(p, \overline{S(\mathcal{L})})\} & \text{if } p \notin \overline{S(\mathcal{L})}, \\ \frac{1}{2} & \text{if } p \in \overline{S(\mathcal{L})}. \end{cases}$$

Recall from the paragraph just before Assertion 3.9 that we found a simple closed curve  $\Gamma_n \subset \widehat{M}_{2,n} \cap \overline{\mathbb{B}}(2)$  which is homotopically non-trivial in  $\widehat{M}_{2,n}$  and disjoint from  $\mathbb{B}(p, 1/2)$  (in

particular,  $\Gamma_n \cap \overline{B_{1,n}} = \emptyset$  for  $n$  large). For  $n$  large, each of the surfaces  $M_{2,n} \cap [\overline{\mathbb{B}(3)} - \mathbb{B}(p, \varepsilon)]$  contains a main planar domain component  $C_n$ , which contains the curve  $\Gamma_n$  and with a long, connected, double spiral curve on  $\partial\mathbb{B}(3)$ . The component  $C_n$  intersects  $\partial\mathbb{B}(p, \varepsilon)$  in either a double spiral curve when  $p \in \overline{S(\mathcal{L})}$  or in a large number of almost-horizontal closed curves when  $p \notin S(\mathcal{L})$ . It follows that  $\widehat{M}_{2,n} \cap \overline{\mathbb{B}(3)}$  consists entirely of disks, which contradicts the existence of  $\Gamma_n$ . This contradiction completes the proof of Lemma 3.6.  $\square$

**Lemma 3.10** *The sequence  $\{M_{2,n}\}_n$  is locally simply connected in  $\mathbb{R}^3$ .*

*Proof.* The proof of this lemma is similar to the proof of Assertion 3.7. The failure of  $\{M_{2,n}\}_n$  to be locally simply connected at a point  $q \in \mathbb{R}^3$  implies for  $n$  large the existence of a disk  $D_n \subset \widehat{M}_{2,n}$  whose boundary curve  $\Gamma_n \subset M_{2,n}$  is contained in a ball  $\mathbb{B}(q, \varepsilon)$  of small radius  $\varepsilon \in (0, 1)$  ( $\Gamma_n$  is homotopically non-trivial in  $M_{2,n}$ ), and such that  $D_n$  intersects  $B_{1,n}$ .

Since  $r_{2,n} \geq 1$  as proven in the paragraph just before (4), then the radius of  $B_{1,n}$  is less than or equal to  $R_1$ . As  $B_{1,n}$  leaves every compact set for  $n$  large, we deduce from the maximum principle that there is no connected minimal surface with boundary contained in  $\mathbb{B}(q, \varepsilon) \cap B_{1,n}$  when the distance between the balls  $\mathbb{B}(q, \varepsilon)$ ,  $B_{1,n}$  is sufficiently large. This contradicts the existence of the minimal planar domain  $D_n - B_{1,n}$ , and completes the proof of the lemma.  $\square$

Applying Theorem 2.2 to the sequence  $\{M_{2,n}\}_n$  with  $W = \emptyset$ , we conclude that there exists a minimal lamination  $\mathcal{L}_2$  of  $\mathbb{R}^3$  and a closed subset  $S(\mathcal{L}_2) \subset \mathcal{L}_2$  such that after passing to a subsequence, the  $M_{2,n}$  converge in  $\mathbb{R}^3 - S(\mathcal{L}_2)$  to  $\mathcal{L}_2$ . Now our previous arguments in Lemmas 3.1, 3.2 and 3.4 apply without modifications and complete the proof of the following proposition.

**Proposition 3.11** *After passing to a subsequence, the  $M_{2,n}$  converge smoothly with multiplicity 1 to a connected, properly embedded minimal surface  $L_2 \subset \mathbb{R}^3$  that lies in one of the cases (C1) or (C2) stated at the end of Section 3.3.*

### 3.7 Surgery in the second stage.

With the notation of the previous proposition and given  $R_2 > 0$ , we let  $L_2(R_2) = L_2 \cap \overline{\mathbb{B}(R_2)}$ , where the radius  $R_2$  is chosen large enough so that  $L_2 - L_2(R_2)$  consists of annular representatives of the ends of  $L_2$ . As we did in replacements (E1), (E2) in Section 3.4, we perform the corresponding replacement of  $L_2(R_2)$  by disks to obtain a smooth, properly embedded (nonminimal) surface  $\tilde{L}_2 \subset \mathbb{R}^3$ . Since  $\{M_{2,n}\}_n$  converges smoothly with multiplicity 1 to  $L_2$  and  $B_{1,n}$  leaves any compact set of  $\mathbb{R}^3$  for  $n$  large enough, then the sequence  $\{\widehat{M}_{2,n}\}_n$  also converges smoothly with multiplicity 1 to  $L_2$ .

Finally, replace  $\widehat{M}_{2,n} \cap \overline{\mathbb{B}(R_2)}$  in a similar way to get a new smooth properly embedded surface  $\widetilde{M}_{2,n} \subset \mathbb{R}^3$  which is not minimal but is  $C^2$ -close to  $\tilde{L}_2$  in  $\overline{\mathbb{B}(R_2)}$ . Note that  $\widetilde{M}_{2,n}$  has simpler topology than  $M_{2,n}$ , the simplification of topology occurring as two replacements by collections of disks inside the balls  $B_{1,n}$  and  $\mathbb{B}(R_2)$ . This finishes the second stage in a recursive definition of properly embedded surfaces obtained as rescalings and disk replacements from the original surfaces  $M(n)$ .

### 3.8 The inductive process.

We now proceed inductively to produce the  $k$ -th stage. After passing to a subsequence of the original surfaces  $M(n)$ , we assume that for each  $i < k$  the following properties hold:

- (F1) There exist largest numbers  $r_{i,n} \geq 1$  such that in every open ball  $B \subset \mathbb{R}^3$  of radius  $r_{i,n}$ , every simple closed curve in  $\widetilde{M}_{i-1,n} \cap B$  bounds a disk contained in  $\widetilde{M}_{i-1,n}$  (in the case  $i = 1$  we let  $\widetilde{M}_{0,n}$  to be  $M(n)$ ).
- (F2) There exist points  $T_{i,n} \in \mathbb{R}^3$  such that  $\widetilde{M}_{i-1,n} \cap \overline{\mathbb{B}}(T_{i,n}, r_{i,n})$  contains a simple closed curve that is homotopically non-trivial in  $\widetilde{M}_{i-1,n}$ .
- (F3) The sequence of surfaces  $M_{i,n} = \frac{1}{r_{i,n}}(M_{i-1,n} - T_{i,n})$  is locally simply connected in  $\mathbb{R}^3$ , all being rescaled images of the original surfaces  $M(n)$ .
- (F4)  $\{M_{i,n}\}_n$  converges smoothly with multiplicity 1 to a connected, properly embedded minimal surface  $L_i \subset \mathbb{R}^3$  that lies in one of the cases (C1) or (C2) stated at the end of Section 3.3.
- (F5) The surface  $\widehat{M}_{i,n} = \frac{1}{r_{i,n}}(\widetilde{M}_{i-1,n} - T_{i,n})$  has simpler topology than  $M_{i,n}$ : the simplification of the topology of  $M_{i,n}$  giving  $\widehat{M}_{i,n}$  consists of  $i - 1$  replacements by collections of disks and these replacements occur in  $i - 1$  disjoint balls that leave each compact set of  $\mathbb{R}^3$  as  $n \rightarrow \infty$ . Furthermore,  $M_{i,n}, \widehat{M}_{i,n}$  coincide outside such  $i - 1$  balls.
- (F6) There exists a large number  $R_i > 0$  such that  $L_i - L_i(R_i)$  consists of annular representatives of the ends of  $L_i$ .
- (F7) There exists a smooth, properly embedded (nonminimal) surface  $\widetilde{L}_i \subset \mathbb{R}^3$  such that  $\widetilde{L}_i$  coincides with  $L_i$  in  $\mathbb{R}^3 - \mathbb{B}(R_i)$  and  $\widetilde{L}_i \cap \overline{\mathbb{B}}(R_i)$  is either a disk (when  $L_i$  is in case (C1)) or a finite number of almost-flat disks (if  $L_i$  is in case (C2)).
- (F8) There exist smooth, properly embedded (nonminimal) surfaces  $\widetilde{M}_{i,n} \subset \mathbb{R}^3$  such that  $\widetilde{M}_{i,n}$  coincides with  $\widehat{M}_{i,n}$  in  $\mathbb{R}^3 - \mathbb{B}(R_i)$  and  $\widetilde{M}_{i,n} \cap \overline{\mathbb{B}}(R_i)$  is arbitrarily  $C^2$ -close to  $\widetilde{L}_i \cap \overline{\mathbb{B}}(R_i)$ . Note that by (F5) above, the surface  $\widetilde{M}_{i,n} \cap \overline{\mathbb{B}}(R_i)$  coincides with  $M_{i,n} \cap \overline{\mathbb{B}}(R_i)$ . As a consequence,  $\widetilde{M}_{i,n}$  has simpler topology than  $M_{i,n}$ , with the simplification of topology consisting of  $i$  replacements by collections of disks, one of these replacements occurring in  $\overline{\mathbb{B}}(R_i)$  and the remaining ones inside  $i - 1$  balls that leave each compact set of  $\mathbb{R}^3$  as  $n \rightarrow \infty$ . Outside these  $i$  balls,  $\widetilde{M}_{i,n}$  and  $M_{i,n}$  coincide.

We now describe how to define  $r_{k,n}, T_{k,n}, M_{k,n}, L_k, \widehat{M}_{k,n}, R_k, \widetilde{L}_k$  and  $\widetilde{M}_{k,n}$ .

We define  $r_{k,n}$  to be the largest positive number such that for every open ball  $B \subset \mathbb{R}^3$  of radius  $r_{k,n}$ , every simple closed curve in  $\widetilde{M}_{k-1,n} \cap B$  bounds a disk in  $\widetilde{M}_{k-1,n}$ . Following the arguments in the second paragraph of Section 3.5, one proves that  $r_{k,n} \geq 1$ . Furthermore, there exists a closed ball of radius  $r_{k,n}$  centered at a point  $T_{k,n} \in \mathbb{R}^3$  whose intersection with  $\widetilde{M}_{k-1,n}$  contains a simple closed curve that is homotopically non-trivial in  $\widetilde{M}_{k-1,n}$ . We denote by

$$M_{k,n} = \frac{1}{r_{k,n}}(M_{k-1,n} - T_{k,n}), \quad \widehat{M}_{k,n} = \frac{1}{r_{k,n}}(\widetilde{M}_{k-1,n} - T_{k,n}).$$

Hence,  $M_{k,n}$  is a rescaled and translated image of the original surface  $M(n)$ , and  $\widehat{M}_{k,n} \cap \overline{\mathbb{B}}(1)$  contains a curve that is homotopically non-trivial in  $\widehat{M}_{k,n}$ . Finally,  $\widetilde{M}_{k,n}$  is obtained from  $M_{k,n}$  after  $k - 1$  replacements by collections of disks, one of these replacements occurring inside the ball

$$B_{k-1,n} = \frac{1}{r_{k,n}}(\mathbb{B}(R_{k-1}) - T_{k,n})$$

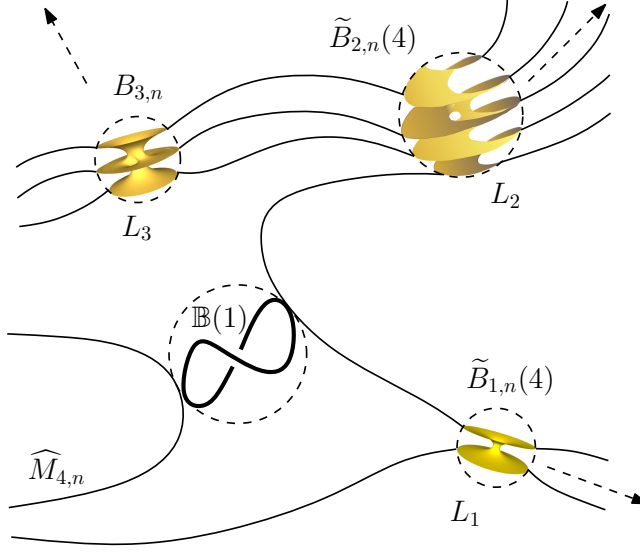


Figure 4: The inductive process in the stage  $k = 4$ .

and the remaining  $k-2$  replacements in pairwise disjoint balls  $\tilde{B}_{1,n}(k), \dots, \tilde{B}_{k-2,n}(k)$  that are disjoint from  $B_{k-1,n}$  and where rescaled and translated images of the forming limits  $L_1, \dots, L_{k-2}$  are captured (see Figure 4). Note that the radius of  $B_{k-1,n}$  is  $\frac{R_{k-1}}{r_{k,n}} \leq R_{k-1}$ , and repeating this argument we have that the radius of  $\tilde{B}_{j,n}(k)$  is less than or equal to  $R_j$  for each  $j = 1, \dots, k-2$ .

**Lemma 3.12** *Given any compact set  $C \subset \mathbb{R}^3$ ,  $B_{k-1,n} \cup \left( \bigcup_{j=1}^{k-2} \tilde{B}_{j,n}(k) \right)$  is disjoint from  $C$  for  $n$  sufficiently large.*

*Proof.* Assume that the lemma fails for some compact set  $C$ . The arguments in the first paragraph of the proof of Lemma 3.6 can be adapted to show that if after choosing a subsequence, some of the balls in the collection  $\mathcal{B} = \{B_{k-1,n}, \tilde{B}_{1,n}(k), \dots, \tilde{B}_{k-2,n}(k)\}$  stay at bounded distance from the origin as  $n$  goes to  $\infty$ , then their corresponding radii go to zero. This implies that, after extracting a subsequence,  $\bigcup \mathcal{B} = B_{k-1,n} \cup \left( \bigcup_{j=1}^{k-2} \tilde{B}_{j,n}(k) \right)$  has non-empty limit set as  $n \rightarrow \infty$  being a finite set of points in  $\mathbb{R}^3$ , denoted by  $\{p_1, \dots, p_l\}$ .

**Assertion 3.13**  *$\{M_{k,n}\}_n$  is locally simply connected in  $\mathbb{R}^3 - \{p_1, \dots, p_l\}$ .*

*Proof.* The proof of the similar fact in Assertion 3.7 does not work in this setting, so we will give a different proof. Arguing by contradiction, we may assume that there exists a point  $p \in \mathbb{R}^3 - \{p_1, \dots, p_l\}$  such that for any  $r \in (0, 1)$  fixed and for  $n$  sufficiently large, there exists a homotopically non-trivial curve  $\gamma_{k,n}(r)$  in  $M_{k,n} \cap \mathbb{B}(p, r)$ . By our normalization,  $\gamma_{k,n}(r)$  bounds a disk  $\hat{D}_{k,n}(r)$  on  $\widehat{M}_{k,n}$ . Since  $\widehat{M}_{k,n}$  and  $M_{k,n}$  coincide outside  $\bigcup \mathcal{B}$ , we deduce from the convex hull property that  $\hat{D}_{k,n}(r)$  must enter some of the balls in  $\mathcal{B}$ . Hence,  $\hat{D}_{k,n}(r)$  intersects the boundary of  $\bigcup \mathcal{B}$  in a non-empty collection  $\mathcal{A}$  of curves, each of which is arbitrarily close to a rescaled and translated image of the intersection of a sphere of large

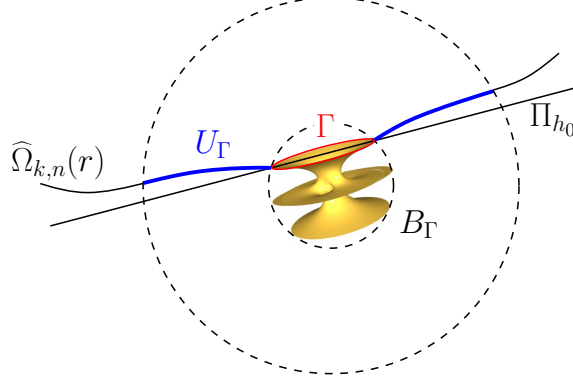


Figure 5:  $\widehat{\Omega}_{k,n}(r)$  lies outside  $B_\Gamma$  along  $\Gamma$ .

radius centered at the origin with either an embedded minimal end of finite total curvature or with a helicoidal end. Define

$$\Omega_{k,n}(r) = \widehat{D}_{k,n}(r) - \bigcup \mathcal{B}$$

which is a planar domain in  $M_{k,n}$  whose boundary consists of  $\gamma_{k,n}(r)$  together with the curves in  $\mathcal{A}$ . Let  $\widehat{\Omega}_{k,n}(r)$  be the compact subdomain on  $M_{k,n}$  obtained by gluing to  $\Omega_{k,n}(r)$  the forming helicoids with handles whose boundary curves lie on  $\Omega_{k,n}(r)$ . Since  $\widehat{\Omega}_{k,n}(r)$  is a compact minimal surface with boundary and  $\bigcup \mathcal{B}$  is disjoint from  $\mathbb{B}(p, r)$  for  $n$  large, then the convex hull property implies that  $\widehat{\Omega}_{k,n}(r)$  has at least one boundary curve  $\Gamma$  outside  $\mathbb{B}(p, r)$ . Then, the following properties hold.

- (G1)  $\Gamma$  lies in the boundary of one of the balls in  $\mathcal{B}$ , which we simply denote by  $B_\Gamma$ . Observe that  $\Gamma$  is the intersection of the boundary of  $\Omega_{k,n}(r)$  and the boundary of a compact domain  $\Omega' \subset M_{k,n}$  obtained as intersection of  $M_{k,n}$  with the closure of  $B_\Gamma$ . Furthermore,  $\Omega'$  is a compact domain in  $M_{k,n}$  where a replacement of type (E2) has been made in a stage previous to the  $k$ -stage.
- (G2) There exists a closed neighborhood  $U_\Gamma$  of  $\Gamma$  in  $\widehat{\Omega}_{k,n}(r)$  that lies outside  $B_\Gamma$ , an end  $\widetilde{E}$  of a vertical catenoid centered at the origin  $\vec{0} \in \mathbb{R}^3$  or of the plane  $\{x_3 = 0\}$  and a map  $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , which is the composition of a homothety and a rigid motion, such that  $U_\Gamma$  can be taken arbitrarily close to  $\phi(E)$  (by taking  $n$  large enough and the radii  $R_i$ ,  $i < k$ , fixed but sufficiently large), where  $E$  is the intersection of  $\widetilde{E}$  with the closed region between two spheres of large radii centered at  $\vec{0}$ , so that  $\Gamma$  corresponds through  $\phi$  to the intersection of  $E$  with the inner sphere, see Figure 5.

In particular, the normal vector to  $\widehat{\Omega}_{k,n}(r)$  along  $\Gamma$  can be assumed to lie in an arbitrarily small open disk in the unit sphere, centered at the image by the linear part of  $\phi$  of the limit normal vector to  $\widetilde{E}$ . In the case  $\widetilde{E}$  is the end of a catenoid, the compact subdomain  $E$  can be chosen as the intersection of  $\widetilde{E}$  with a slab of the type  $\{(x_1, x_2, x_3) \mid a \leq x_3 \leq b\}$ , for  $0 < a < b$  large. For  $a$  fixed and  $b > a$  arbitrarily large, the sublinearity of the growth of the third coordinate function on  $\widetilde{E}$  implies that if a plane  $\Pi_1 \subset \mathbb{R}^3$  touches  $E$  at a point of  $\{x_3 = a\}$  and leaves  $E$  at one side of  $\Pi_1$ , then  $\Pi_1$  must be arbitrarily close to the horizontal in terms of  $a$ . Therefore, if a plane  $\Pi \subset \mathbb{R}^3$  touches  $\widehat{\Omega}_{k,n}(r)$  along  $\Gamma$  and leaves  $\widehat{\Omega}_{k,n}(r)$  at one

side of  $\Pi$ , then the orthogonal direction to  $\Pi$  must be arbitrarily close to  $\pm\phi(0, 0, 1)$ . A similar conclusion holds when  $\bar{E}$  is the end of the plane  $\{x_3 = 0\}$ .

Since the number of components of  $\partial\hat{\Omega}_{k,n}(r) - \mathbb{B}(p, r)$  is bounded independently of  $n$ , we deduce that the normal lines to  $\hat{\Omega}_{k,n}(r)$  along its boundary curves other than  $\gamma_{k,n}(r)$  lie in a collection  $\mathcal{D}_n$  of arbitrarily small (for  $n$  sufficiently large) open disks in the projective plane  $\mathbb{P}^2$ , the number of which is bounded independently of  $n$ . Furthermore, as  $n \rightarrow \infty$ , the  $\mathcal{D}_n$  converge after extracting a subsequence to a finite set  $\mathcal{D}_\infty \subset \mathbb{P}^2$ . Given  $n$  large, consider a furthest point  $q_n$  in  $\partial\hat{\Omega}_{k,n}(r)$  to  $p$ . Let  $F \in \mathbb{P}^2$  be the line obtained as a (subsequential) limit of the directions of the position vectors  $q_n - p$ . Suppose for the moment that  $F$  does not lie in  $\mathcal{D}_\infty$ . Consider the family of planes  $\{\Pi_h \subset \mathbb{R}^3 \mid h \in \mathbb{R}\}$  orthogonal to  $F$ ,  $h$  being the oriented distance to  $p$ . If  $h_0(n) = \text{dist}(p, q_n)$ , then the domain  $\hat{\Omega}_{k,n}(r)$  lies entirely at one side of the plane  $\Pi_{h_0(n)}$ . Taking  $n \rightarrow \infty$ , the argument in the last paragraph implies that  $F$  lies in  $\mathcal{D}_\infty$ , which is contrary to our assumption. Hence  $F \in \mathcal{D}_\infty$ . Fix  $n$  large and consider the plane  $\Pi_{h_0(n)}$  which has  $\hat{\Omega}_{k,n}(r)$  at one side, with  $q_n \in \Pi_{h_0(n)} \cap \hat{\Omega}_{k,n}(r)$ . Fix  $\varepsilon > 0$  small and tilt slightly  $\Pi_{h_0(n)+\varepsilon}$  to produce a new plane  $\Pi'$  so that  $\hat{\Omega}_{k,n}(r)$  still lies at one side of  $\Pi'$  and  $\Pi' \cap \hat{\Omega}_{k,n}(r) = \emptyset$ . Observe that the normal direction to  $\Pi'$  can be assumed not to lie in  $\mathcal{D}_\infty$ . Move  $\Pi'$  towards  $\hat{\Omega}_{k,n}(r)$  by parallel planes until we find a first contact point (which must lie on  $\partial\hat{\Omega}_{k,n}(r)$  by the maximum principle). Now we can repeat the argument as before with  $\Pi'$  instead of  $\Pi_{h_0(n)}$  to find a contradiction if  $n$  is sufficiently large. This proves the assertion.  $\square$

Consider for each  $i = 1, \dots, l$  a ball  $B_i$  centered at  $p_i$ , whose radius is much smaller than the minimum distance between pairs of distinct points  $p_j, p_h$  with  $j, h \in \{1, \dots, l\}$ . To proceed with the proof of Lemma 3.12, we need the following property.

**Assertion 3.14** *For each  $i$ , the components of  $\widehat{M}_{k,n}$  in  $B_i$  are disks for  $n$  large.*

*Proof.* To see this, suppose that for a given  $i = 1, \dots, l$ , there exists a simple closed curve  $\gamma_{k,n} \subset \widehat{M}_{k,n} \cap B_i$  that does not bound a disk on  $\widehat{M}_{k,n} \cap B_i$ . As the radius of  $B_i$  can be assumed to be less than 1, then  $\gamma_{k,n}$  must bound a disk in  $\widehat{M}_{k,n}$ . Now the same proof of Assertion 3.13 gives a contradiction, thereby proving Assertion 3.14.  $\square$

We next continue with the proof of Lemma 3.12. For the fixed value  $k$ , define the sequence of compact minimal surfaces

$$\left\{ M_{k,n} \cap \left[ \mathbb{B}(n) - \bigcup_{i=1}^l \mathbb{B}(p_i, 1/n) \right] \right\}_n, \quad (5)$$

which has locally positive injectivity radius in  $A = \mathbb{R}^3 - \{p_1, \dots, p_l\}$  by Assertion 3.13. Applying Theorem 2.2 to this sequence with  $W = \{p_1, \dots, p_l\}$ , we deduce that there exists a minimal lamination  $\mathcal{L}$  of  $\mathbb{R}^3 - \{p_1, \dots, p_l\}$  and a closed subset  $S(\mathcal{L}) \subset \mathcal{L}$  such that the sequence defined by (5) converges  $C^\alpha$  for all  $\alpha \in (0, 1)$  to  $\mathcal{L}$ , outside of the singular set of convergence  $S(\mathcal{L})$ . Furthermore, the closure  $\bar{\mathcal{L}}$  of  $\mathcal{L}$  has the structure of a minimal lamination of  $\mathbb{R}^3$ . In fact, the arguments in the proof of Assertions 3.8 and 3.9 can be easily adapted to demonstrate that  $\bar{\mathcal{L}}$  consists of planes and  $S(\mathcal{L}) \neq \emptyset$ . Therefore, items 2, 3 of Theorem 2.2 imply that  $\bar{\mathcal{L}}$  is a foliation of  $\mathbb{R}^3$  by planes and the convergence of the sequence defined in (5) to  $\bar{\mathcal{L}}$  has the structure of a limit minimal parking garage structure with  $\bar{S}(\mathcal{L})$  consisting of one or two columns. Once we know this, the arguments in the proof of Assertion 3.9 ensure that  $\bar{S}(\mathcal{L})$  intersects  $\mathbb{B}(3)$ .

Again, the arguments in the proof of Lemma 3.4 remain valid and imply that  $\overline{S(\mathcal{L})}$  consists of a single line. The final step in the proof of Lemma 3.12 is to discard the case that  $\overline{S(\mathcal{L})}$  is one line. Consider the positive number

$$\varepsilon = \frac{1}{3} \min[\{1\} \cup \{d_{\mathbb{R}^3}(p_i, p_j)\}_{i \neq j}].$$

Arguing as in the last paragraph of the proof of Assertion 3.9 we conclude that there exists a simple closed curve  $\Gamma_n \subset \widehat{M}_{k,n} \cap \overline{\mathbb{B}}(2)$  which is homotopically non-trivial in  $\widehat{M}_{k,n}$  and disjoint from  $\bigcup_{i=1}^l \overline{\mathbb{B}}(p_i, \varepsilon)$ . Furthermore, for  $n$  large each of the surfaces  $M_{k,n} \cap [\overline{\mathbb{B}}(3) - \bigcup_{i=1}^l \overline{\mathbb{B}}(p_i, \varepsilon)]$  contains a main planar domain component  $C_n$ , which contains the curve  $\Gamma_n$ . For  $n$  large,  $C_n$  intersects  $\partial \mathbb{B}(3)$  in a long, connected, double spiral curve, and it intersects  $\partial \mathbb{B}(p_i, \varepsilon)$  in either a double spiral curve when  $p_i \in \overline{S(\mathcal{L})}$  or in a large number of almost-horizontal closed curves when  $p_i \notin \overline{S(\mathcal{L})}$ ,  $i = 1, \dots, l$ . Thus,  $\widehat{M}_{k,n} \cap \overline{\mathbb{B}}(3)$  consists of disks, which contradicts the existence of  $\Gamma_n$ . Now the proof of Lemma 3.12 is complete.  $\square$

Straightforward modifications in the proof of Lemmas 3.10 and Proposition 3.11 give the following lemma, whose proof we leave to the reader.

**Lemma 3.15** *The sequence  $\{M_{k,n}\}_n$  is locally simply connected in  $\mathbb{R}^3$ , and after passing to a subsequence, it converges with multiplicity 1 to a minimal lamination  $\mathcal{L}_k$  of  $\mathbb{R}^3$  consisting of a single leaf  $L_k$  which satisfies the properties in cases (C1) or (C2).*

We can continue this inductive process indefinitely and using a diagonal subsequence, we will obtain an infinite sequence  $\{L_k\}_{k \in \mathbb{N}}$  of non-simply connected, properly embedded minimal surfaces, each one satisfying one of the properties (C1) or (C2). For each fixed  $k \in \mathbb{N}$ ,  $L_k \cap \mathbb{B}(R_k)$  is the limit under homotheties and translations of compact domains of  $M(n)$  that are contained in balls  $\widehat{B}_{n,k}$ . Moreover,  $\widehat{B}_{n,k}$  is disjoint from  $\widehat{B}_{n,k'}$  for  $k \neq k'$ .

### 3.9 The final contradiction.

Since the genus of  $M(n)$  is fixed and finite, for all  $k$  sufficiently large the surface  $L_k$  has genus zero (in particular,  $L_k$  cannot satisfy case (C1), see also the caption of Figure 3). By the López-Ros Theorem [26],  $L_k$  is a catenoid. Fix  $k_0$  such that for every  $k \geq k_0$ ,  $L_k$  is a catenoid. Given  $k \geq k_0$ , the convergence to  $L_k$  of suitable homotheties and translations of compact domains of the  $M(n)$  contained in the balls  $\widehat{B}_{n,k}$  appearing in the last paragraph ensures that there exists an integer  $n(k)$  such that for all  $n \geq n(k)$ , we may assume that  $M(n) \cap \widehat{B}_{n,k_0}, \dots, M(n) \cap \widehat{B}_{n,k}$  are close to  $k - k_0 + 1$  catenoids. For these  $k, n$  and for any integer  $k'$  with  $k_0 \leq k' \leq k$ , let  $\Gamma_{n,k'}$  be the unique simple closed geodesic in  $M(n) \cap \widehat{B}_{n,k'}$  which, after scaling and translation, converges to the waist circle of  $L_{k'}$  as  $n \rightarrow \infty$ .

**Lemma 3.16** *For any  $m \in \mathbb{N}$ , there exists  $k \geq k_0$  such that at least  $m$  of the simple closed curves  $\Gamma_{n(k),k'} \subset M(n(k)) \cap \widehat{B}_{n(k),k'}$  separate  $M(n(k))$  where  $k_0 \leq k' \leq k$ .*

*Proof.* If the lemma were to fail, then for any  $k \geq k_0$ , there would be a bound on the number of the curves  $\Gamma_{n(k),k'}$  that separate  $M(n(k))$ . Since the genus of  $M(n(k))$  is independent of  $k$ , for  $k$  sufficiently large there exist seven of these geodesics  $\Gamma_{n(k),k'}$  that bound two consecutive annuli in the conformal compactification  $\overline{M}(n(k))$  of  $M(n(k))$ . More precisely, we find  $\Lambda_1, \Gamma_1, \Lambda_2, \Gamma_2, \Lambda_3, \Gamma_3, \Lambda_4$ , seven of the non-separating curves  $\Gamma_{n(k),k'}$ , so that

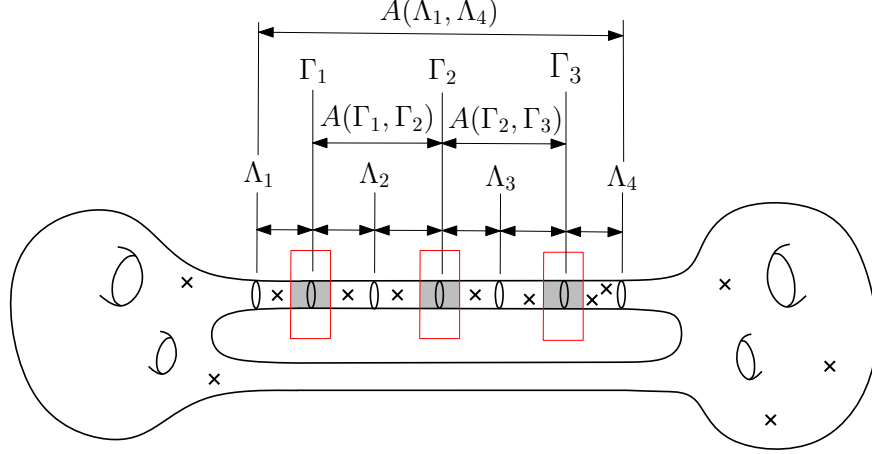


Figure 6: Properties (H1), (H2): Topological representation of  $M(n(k))$ , with ends represented by crosses. When viewed in  $\mathbb{R}^3$ , the compact portions of  $M(n(k))$  enclosed in the red rectangles represent almost perfectly formed catenoids, whose almost waist circles are the non-separating curves  $\Gamma_i$ .

(H1)  $\Lambda_1 \cup \Lambda_4$  is the boundary of a compact annulus  $A(\Lambda_1, \Lambda_4) \subset \overline{M}(n(k))$  that is the union of compact annuli  $A(\Lambda_1, \Gamma_1)$ ,  $A(\Gamma_1, \Lambda_2)$ ,  $A(\Lambda_2, \Gamma_2)$ ,  $A(\Gamma_2, \Lambda_3)$ ,  $A(\Lambda_3, \Gamma_3)$ ,  $A(\Gamma_3, \Lambda_4)$  that do not intersect in their interiors. Observe by the convex hull property, all of these annuli contain at least 1 end of  $M(n(k))$ , as does  $M(n(k)) - A(\Lambda_1, \Lambda_4)$ , see Figure 6.

Let  $A(\Gamma_1, \Gamma_2)$ ,  $A(\Gamma_2, \Gamma_3)$  be the related subannuli in  $\overline{M}(n(k))$  bounded by  $\Gamma_1 \cup \Gamma_2$ ,  $\Gamma_2 \cup \Gamma_3$ , respectively.

Observe that (H1) now implies:

(H2) Each of the three components of  $M(n(k)) - (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$  contains at least two ends of  $M(n(k))$ .

**Assertion 3.17** *In the above situation,  $\Gamma_1, \Gamma_2, \Gamma_3$  all bound disks on the same closed complement of  $M(n(k))$  in  $\mathbb{R}^3$ .*

*Proof.* Suppose that the assertion fails. Then, we may assume without loss of generality that  $\Gamma_1$  and  $\Gamma_2$  bound disks on opposite sides of  $M(n(k))$ . Let  $W, W_1$  denote the closures of the two components of  $\mathbb{R}^3 - M(n(k))$ , so that  $\Gamma_1$  bounds a disk  $D_1$  in  $W_1$  and  $\Gamma_2$  bounds another disk  $D_2$  in  $W$ . Observe that  $\Gamma_1$  is not homologous to zero in  $W$  (otherwise  $\Gamma_1$  would separate  $M(n(k))$ , which is contrary to our hypothesis). Let  $\Omega \subset M(n(k))$  be the (noncompact) planar domain bounded by  $\Gamma_1 \cup \Gamma_2$ . After a small perturbation of  $\Omega \cup D_2$  in  $W$  that fixes  $\Gamma_1$ , we obtain a new surface  $\Sigma$  contained in  $W$ , such that  $\Sigma \cap M(n(k)) = \Gamma_1$ . The union of  $\Sigma$  together with  $D_1$  is a properly embedded surface that intersects  $M(n(k))$  only along  $\Gamma_1$ . This implies that  $\Gamma_1$  separates  $M(n(k))$ , which is a contradiction.  $\square$

Let  $W_1$  be the closure of the component of  $\mathbb{R}^3 - M(n(k))$  in which  $\Gamma_1, \Gamma_2, \Gamma_3$  all bound disks, which exists by Assertion 3.17. Since none of the  $\Gamma_1, \Gamma_2, \Gamma_3$  separate  $M(n(k))$ , then none of the  $\Gamma_1, \Gamma_2, \Gamma_3$  bound properly embedded surfaces in the closure  $W$  of  $\mathbb{R}^3 - W_1$ . As



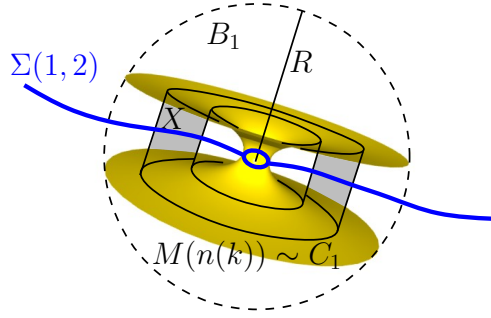


Figure 7: The normal line to  $\Sigma(1, 2) \cap X$  is close to the limit normal line to  $C_1$ .

$\Gamma_1 \cup \Gamma_2 \subset \partial W$  bounds a connected, non-compact orientable surface in  $W$  (which is part of  $M(n(k))$ ) and  $\partial W$  is a good barrier for solving Plateau problems in  $W$ , a standard argument [38, 39] ensures that there exists a non-compact, connected, orientable least-area surface  $\Sigma(1, 2) \subset W$  with boundary  $\partial\Sigma(1, 2) = \Gamma_1 \cup \Gamma_2$ .

**Assertion 3.18**  $\Sigma(1, 2)$  has just one end, this end has vertical limiting normal vector, and  $\Sigma(1, 2) \cap \partial W = \Gamma_1 \cup \Gamma_2$ .

*Proof.* Recall that the simple closed curves  $\Gamma_1, \Gamma_2, \Gamma_3$  are the unique closed geodesics in the intersection of  $M(n(k))$  with disjoint balls  $B_1, B_2, B_3$  and that  $M(n(k)) \cap B_i$  can be assumed to be arbitrarily close to a large region of a catenoid  $C_i$  centered at the center of  $B_i$  (and suitably rescaled),  $i = 1, 2, 3$ . In order to check that  $\Sigma(1, 2)$  has exactly one end, let  $X$  be the non-simply connected region of  $B_1 - M(n(k))$  that lies between two coaxial cylinders with axis the axis of  $C_1$  and radii  $\frac{R}{3}, \frac{R}{2}$  where  $R$  denotes the radius of  $B_1$ , see Figure 7.

By curvature estimates for stable surfaces, the portion of  $\Sigma(1, 2)$  contained in  $X$  consists of almost-flat graphs parallel to the almost-flat graphs defined by the catenoid  $C$  in the boundary of  $X$ . Since the surface  $\Sigma(1, 2)$  is area-minimizing in  $W$ , there is only one such an annular graph. A similar description can be made for  $\Sigma(1, 2)$  in the ball  $B_2$ . After removing the portion of  $\Sigma(1, 2)$  inside the innermost cylinder in each of these balls, we obtain a connected, non-compact, stable minimal surface  $\tilde{\Sigma}(1, 2)$  whose Gauss map  $\tilde{G}: \tilde{\Sigma}(1, 2) \rightarrow \mathbb{S}^2$  satisfies that  $\tilde{G}(\partial\tilde{\Sigma}(1, 2))$  is contained in a small neighborhood of the limiting normal directions of the corresponding forming catenoids  $C_1, C_2$ . Since the surface  $\tilde{\Sigma}(1, 2)$  is stable and connected, it follows that  $\tilde{G}(\tilde{\Sigma}(1, 2))$  is contained in a small neighborhood  $U$  of a point in  $\mathbb{S}^2$  (see a similar argument in the penultimate paragraph of the proof of Lemma 3.2). In particular, the two forming catenoids in  $B_1, B_2$  are almost-parallel. Since  $\tilde{\Sigma}(1, 2)$  lies in the complement of  $M(n(k))$ , then the values of  $\tilde{G}$  at the ends of  $\tilde{\Sigma}(1, 2)$  are the North or the South poles, which are contained in  $U$ . Thus, the forming catenoids inside  $B_1, B_2$  are approximately vertical and  $\tilde{\Sigma}(1, 2)$  is an almost-horizontal graph over its projection to the  $(x_1, x_2)$ -plane. In particular,  $\Sigma(1, 2)$  has exactly one end and this end has vertical limiting normal vector.

Finally, the property that  $\Sigma(1, 2) \cap \partial W = \Gamma_1 \cup \Gamma_2$  can be easily deduced from the maximum principle and from property (H2) above. This completes the proof of the assertion.  $\square$

Note that  $\Sigma(1, 2)$  separates  $W$  into two regions. Let  $W'$  be the closed complement of  $\Sigma(1, 2)$  in  $W$  that contains  $\Gamma_3$  in its boundary. Let  $\Gamma'_2 \subset M(n(k)) \cap B_2$  be an  $\varepsilon$ -parallel

curve to  $\Gamma_2$  in  $\partial W'$ , with  $\varepsilon > 0$  being very small. Since  $\Gamma'_2 \cup \Gamma_3$  bounds a connected non-compact surface in  $\partial W'$  (which is part of  $M(n(k))$ ), then  $\Gamma'_2 \cup \Gamma_3$  also bounds a connected, non-compact, orientable least-area surface  $\Sigma(2, 3) \subset W'$ . Note that the arguments in the proof of Assertion 3.18 apply to give that  $\Sigma(2, 3)$  intersects  $\partial W'$  only along  $\Gamma'_2 \cup \Gamma_3$ , and that outside  $B_2, B_3$ , the surface  $\Sigma(2, 3)$  is an almost-flat, almost-horizontal graph over its projection to the  $(x_1, x_2)$ -plane.

Let  $\Sigma$  be the connected, non-compact, piecewise smooth surface consisting of

$$\Sigma = \Sigma(1, 2) \cup \Sigma(2, 3) \cup D_1 \cup D_3 \cup A(\Gamma_2, \Gamma'_2),$$

where for  $i = 1, 3$ ,  $D_i$  is a disk in  $\mathbb{R}^3 - W$  bounded by  $\Gamma_i$  and  $A(\Gamma_2, \Gamma'_2) \subset M(n(k)) \cap B_2$  is the compact annulus bounded by  $\Gamma_2 \cup \Gamma'_2$ . Since  $\Sigma$  has no boundary and is properly embedded in  $\mathbb{R}^3$ , then  $\Sigma$  separates  $\mathbb{R}^3$  into two open regions. Let  $\Delta(1, 3)$  be the connected component of  $M(n(k)) - (\Gamma_1 \cup \Gamma_3)$  that is disjoint from  $\Gamma_2$ . Let  $W''$  be the connected component of  $\mathbb{R}^3 - \Sigma$  that contains  $\Delta(1, 3)$ . Note that  $\Delta(1, 3)$  separates  $W''$  (this follows because the piecewise smooth properly embedded surface  $\Delta(1, 3) \cup D_1 \cup D_3$  separates  $\mathbb{R}^3$ , the disks  $D_1, D_3$  are contained in the boundary of  $W''$  and  $\Delta(1, 3)$  is contained in  $W''$ ). Let  $W'''$  be the closed complement of  $\Delta(1, 3)$  in  $W''$  in which  $\Gamma_1$  is not homologous to zero. Let  $\Gamma'_1, \Gamma'_3 \subset M(n(k))$  be  $\varepsilon$ -parallel curves to  $\Gamma_1, \Gamma_3$  in  $\partial W'''$ , with  $\varepsilon > 0$  small. Since neither  $\Gamma'_1$  nor  $\Gamma'_3$  separate  $M(n(k))$ , then  $\Gamma'_1 \cup \Gamma'_3$  bounds a connected non-compact, proper surface in  $\partial W'''$  which is part of  $M(n(k))$ . Thus,  $\Gamma'_1 \cup \Gamma'_3$  also bounds a connected, orientable, non-compact, properly embedded least-area surface  $\Sigma(1, 3) \subset W'''$ . As in Assertion 3.18,  $\Sigma(1, 3)$  has exactly one end that is an almost-horizontal graph. The end of this graph lies between the ends of the two horizontal annular ends of  $\Sigma$  since it lies in  $W''' \subset W''$ , see Figure 8.

We now obtain the desired contradiction. Consider the surface

$$\tilde{\Sigma}(1, 3) = \Sigma(1, 3) \cup D'_1 \cup D'_3,$$

where  $D'_i$  is a disk in  $\mathbb{R}^3 - W$  bounded by  $\Gamma'_i$ ,  $i = 1, 3$ . The surface  $\tilde{\Sigma}(1, 3)$  is properly embedded in  $\mathbb{R}^3$  and  $\tilde{\Sigma}(1, 3) \cap \Sigma = \emptyset$ , hence,  $\Sigma$  must lie on one side of  $\tilde{\Sigma}(1, 3)$  in  $\mathbb{R}^3$ . However, since the graphical end of  $\tilde{\Sigma}(1, 3)$  lies between two graphical ends of  $\Sigma$ , we obtain a contradiction that finishes the proof of Lemma 3.16.  $\square$

We now complete the proof of Theorem 1.4. By Lemma 3.16, for any  $m$  there exists  $k \geq k_0$  such that at least  $m$  of the  $k - k_0 + 1$  closed geodesics of the type  $\Gamma_{n(k), k'} \subset M(n(k)) \cap \hat{B}_{n(k), k'}$  separate  $M(n(k))$ ,  $k_0 \leq k' \leq k$ . These  $m$  separating curves  $\Gamma_{n(k), k'}$  can be assumed to be arbitrarily close to the waist circles of suitable rescaled, large compact regions of  $m$  disjoint catenoids. In particular,  $M(n(k))$  has non-zero flux vector along any of these curves, and the separating property implies that such flux vectors are all vertical (any separating curve in  $M(n(k))$  with non-zero flux must be homologous to a finite positive number of ends of  $M(n(k))$ , which have vertical flux). Therefore, the  $m$  forming catenoids inside  $M(n(k))$  are all vertical. Now exchange the geodesics  $\Gamma_{n(k), k'}$  by planar horizontal convex curves  $\tilde{\Gamma}_{n(k), k'}$  in  $M(n(k))$ , which can be chosen arbitrarily close to the corresponding  $\Gamma_{n(k), k'}$ . Since the genus of  $M(n(k))$  is fixed and finite, we can take  $m$  large enough so that at least two of these planar curves, say  $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ , bound a non-compact planar domain  $\Omega$  inside  $M(n(k))$  and bound planar horizontal disks in the same complement of  $M(n(k))$  in  $\mathbb{R}^3$ . Since  $\Omega$  has vertical catenoidal and/or planar ends, the López-Ros deformation [26, 43] applies to give the desired contradiction. This finishes the proof of the theorem.

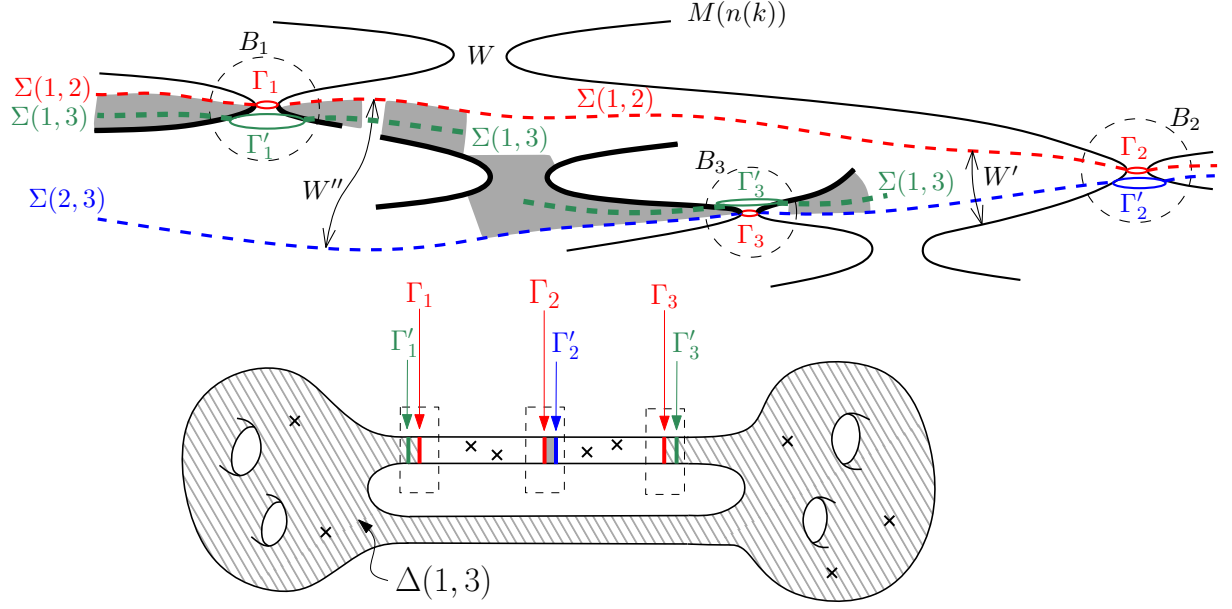


Figure 8: Top: Producing a contradiction with three non-separating curves  $\Gamma_1, \Gamma_2, \Gamma_3 \subset M(n(k))$ . The gray region that contains  $\Sigma(1,3)$  is  $W'''$ . Bottom: A topological representation of  $M(n(k))$ , with the curves that appear in the top figure. The shaded component  $\Delta(1,3)$  of  $M(n(k)) - (\Gamma_1 \cup \Gamma_3)$  in the bottom figure corresponds to the thick black curve in the top figure.

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